On the equivariant motivic spectral sequences

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CONTENTS

Introduction

01. — In the early 1980s, Beilinson and Lichtenbaum independently conjectured the existence of a bi-graded cohomology theory for schemes called *motivic cohomology*, written by

$$X \to H^p(X, \mathbb{Z}(q)).$$

This cohomology theory plays the same role of singular cohomology in topology. It should satisfy various axioms and have certain relation with algebraic K-theory and étale cohomology (c.f [**Bei87**], [**Lic84**]). In particular, Beilinson suggested that for smooth schemes X, there should exist an "Atiyah-Hirzebruch" spectral sequence of the form

$$E_2^{p,q} = H^{p-q}(X, \mathbb{Z}(-q)) \Rightarrow K_{-p-q}(X) \tag{1}$$

relating motivic cohomology and Quillen's algebraic K-theory. This is called *motivic spectral sequence.*

The construction of motivic cohomology as well as the motivic spectral sequence had soon become central problems in the theory of motives and algebraic K-theory during the 1990s and 2000s. Many attempts have been made into this direction.

The first motivic spectral sequence was constructed by Bloch-Lichtenbaum for spectrum of fields. In [**BL95**], they used the idea of multi-relative K-theory with supports together with an assumption of a "rather innocuous looking *moving lemma*". The motivic cohomology groups appeared in Bloch-Lichtenbaum's construction are the *higher Chow groups* constructed by Bloch in [**Blo86**]. The construction and its idea were then extended into two different ways. One extension to smooth varieties over fields was given by Friedlander-Suslin by using another kind of moving lemma (cf. [**FS02**]). The other extension to regular schemes of finite type over Dedekind domain was given by Levine using techniques of localization for algebraic cycles (cf. [**Lev01**]).

Meanwhile, Grayson initiated another construction of motivic spectral sequence for affine regular Noetherian schemes in [**Gra95**] by using a completely different method. His construction uses the K-theory of commuting automorphism associated to vector bundles, following a suggestion of Goodwillie and Lichtenbaum. Grayson's method is elegant and is the *only* known construction for regular (affine) schemes over a general base.

Later on, Voevodsky proposed another approach to construct the motivic spectral sequence for smooth schemes over fields using *motivic homotopy theory.* In [**Voe02a**], he conjectured that the *motivic Postnikov tower* of the spectrum KGL that presents algebraic K-theory are *motivic Eilenberg-MacLane spectra* presenting motivic cohomology.

All of these constructions looks very different from the others and each of which has its own advantages to study. Each construction also gives a candidate for motivic cohomology. It was conjectured that all the above spectral sequences are equivalent. In particular, all the motivic cohomology theories appeared in these motivic spectral sequence were expected to be the same.

The first comparison had been made by Friedlander-Suslin [**FS02**] where they showed that their motivic cohomology agree with higher Chow groups of Bloch. Using this result, Voevodsky then showed that his motivic cohomology are the same with higher Chow groups [**Voe02b**]. These comparisons hold for smooth schemes of finite type over arbitrary fields. The comparison between Grayson's and Voevodsky's motivic cohomology was proved later by Suslin ([**Sus03**]) for semi-local smooth schemes of finite type over fields. However, the coincidence between all these spectral sequences was still open.

Levine, in an attempt to understand the relation between the slice tower and algebraic cycles, has finally proved that the motivic spectral sequences [**BL95**], [**FS02**], [**Lev01**] are equivalent to [**Voe02a**]. In [**Lev08**], Levine used the idea of the topological filtration on cosimplicial schemes to construct the so-called *the homotopy coniveau tower* for S^1- and $\mathbb{P}^1$-spectra, and then showed that his coniveau tower agrees with the slice tower over perfect fields. This allows him to recover and extend a result of Voevodsky on the zeroth layer of the slice tower, hence identify these spectral sequences.

Recently, Garkusha-Panin have proven that the Grayson's and Voevodsky's spectral sequences are equivalent in [**GP12**] by using Suslin's isomorphism between motivic cohomology theories ([**Sus03**]), Levine-Voevodsky's connectedness of motivic cohomology ([**Voe02a**], [**Lev08**]) and Morel's connectivity theorem ([**Mor99**]). Another comparison result appeared in Podkopaev's thesis [**Pod12**], where he showed that the entries in Friedlander-Suslin's and Grayson's towers are equivalent, hence the resulting spectral sequences.

At this moment, we have a solid foundation of the theory of motives for smooth schemes of finite type over fields. In particular, motivic cohomology and motivic spectral sequences are well established in this case. Works of Suslin, Voevodsky, Rost, etc., make motivic cohomology amenable to computation and the motivic spectral sequence is a way where these computations can be used to compute algebraic K-groups, for instance, see Kahn [**Kah02**], Karoubi-Weibel [**KW03**], Rognes-Weibel [**RW00**], Pedrini-Weibel [**PW01**].

02. — By the end of 1980s, in an attempt to prove Seshadri's conjecture on the existence of equivariant resolution, R. Thomason developed a new theory called *equivariant algebraic K-theory* which is a variation of Quillen's algebraic K-theory and Atiyah's equivariant (topological) K-theory (see [**Tho83**]). Let X be a scheme equipped with an action of a (finite, algebraic) group G, the category $\mathcal{P}(G, X)$ of G-vector bundles on X is an *exact category*. The Q-construction of Quillen apply to $\mathcal{P}(G, X)$ to obtain a spectrum $K(G, X)$, where equivariant K-groups $K_n(G, X)$ are defined by taking the nth-homotopy group $\pi_n K(G, X)$. Replacing $\mathcal{P}(G, X)$ by the category $\mathcal{M}(G, X)$ of G-cohenrent sheaves on X we obtain the spectrum $G(G, X)$ and the G-groups $G_n(G, X)$ are $\pi_n G(G, X)$.

Thomason's equivariant K-theory is an interesting invariant in algebraic geometry, arithmetics and representation theory, but as algebraic K-theory, it is very hard to compute in general. Following the success of motivic spectral sequence in understanding algebraic K-theory, it is natural to ask the following questions:

(1) Are there analogies of motivic cohomology and motivic spectral sequence in the equivariant setting?
(2) How do they compare?

03. — One answer for these questions was given by Levine and Serpé in [**LS08**] for smooth schemes of finite type over a field and the order of G is coprime to the characteristic of the base. In the Levine-Serpé's construction, the terms appear in the spectral sequence are the *equivariant higher Chow groups of Bredon type* $CH^p(G, X, q)$ (Definition 2.9). The construction relies on the technique of localization for algebraic cycles (cf. [**Lev01**], [**Lev08**], [**LS08**]) adapted to the equivariant setting.

The equivariant higher Chow groups are interesting objects, they involve to the theory of algebraic cycles with "local coefficient": the representation rings of certain subgroups of G associated to subvarieties. Levine-Serpé also made some computations in the case of codimension 1 (where the 0 codimension case is easy and uninteresting). However, as they pointed out, their construction is *not* contravariantly functorial with respect to G-equivariant morphisms, i.e., if $f: X \to Y$ is a G-equivariant morphism between G-smooth schemes, there might not have a good pull-back $f^*: CH^p(G, Y, *) \to CH^p(G, X, *)$. It is also remarkable that the equivariant higher Chow group is *not* homotopy invariant in general. However, it *does* satisfy homotopy invariant property with respect to the projection $X \times \mathbb{A}^1 \to X$ if one make $X \times \mathbb{A}^1$ a G-scheme via the given action X and the *trivial* action on $\mathbb{A}^1$. There is another phenomenon is the *lack* of the ring of structure on $CH^*(G, X, *)$, even for X smooth. The reason is that the topological filtration on $K(G, X)$ is not functorial with respect to pull back, thus we cannot expect the product on $K_*(G, X)$ to induce a product on $CH^*(G, X, *)$ in the usual way.

04. — In this paper, following Grayson's approach, we establish an equivariant motivic spectral sequence for affine Noetherian regular G-schemes called the *equivariant Grayson spectral sequence*. The construction uses the idea of equivariant K-theory of automorphisms to produce a tower for equivariant K-theory, where the successive layers are weak equivalent to classifying spaces of some simplicial abelian groups. The equivariant Grayson spectral sequence has the form (Corollary 2.6)

$$E_2^{p,q} := H_{Gr}^{p-q}(G, X, -q) \Rightarrow K_{-p-q}(G, X)$$

where $H^p_{Gr}(G, X, q)$ stands for G-equivariant Grayson cohomology group (Definition 2.5).

The Grayson's construction has some interesting properties. It is contravariantly functorial for G-equivariant morphisms. The ring structure on $H^*_{Gr}(G, X, *)$ is naturally induced from the product on $K_*(G, X)$. Moreover, the equivariant Grayson cohomology *does* satisfy homotopy invariance: For given a G-scheme X and a representation V of G on k, the G-equivariant projection $X \times \mathbb{A}(V) \to X$ induces an isomorphism

$$H^p_{Gr}(G, X, q) \cong H^p_{Gr}(G, X \times \mathbb{A}(V), q)$$

where $\mathbb{A}(V)$ denotes the affine (G-) space associated to V (Corollary 3.3).

Unfortunately, Grayson cohomology $H^p_{Gr}(G, X, q)$ involves the direct-sum Grothendieck groups which are very difficult to compute.

05. — To have a better understanding of the two constructions, we will restrict ourself to the the following case: Let K be a field with G action, denote by $k := K^G$ the fixed field of K under the action of G. We assume that K is perfect and the order of G is comprime to the characteristics of K. Consider the two spectral sequences for $Y \times_k K$ where $Y \in \mathbf{Sm}/k$ and G acts on $Y \times_k K$ via its action on K. Denote by $K^{K,G}$ the presheaf of spectrum on $\mathbf{Sm}/k$ which sends Y to $K(G, Y \times_k K)$, we have

***Theorem* (Theorem 5.3).** — *The Levine-Serpé tower for $Y \times_k K$ is the same with the slice tower of $K^{K,G}$ over Y where $Y \in \mathbf{Sm}/k$ is a semi-local smooth scheme of finite type over k.*

In particular, this puts Levine-Serpé's construction into the broader theory of slice tower for motivic spectra.

Using the universal property of the slice tower, we show that

***Theorem* (Theorem 5.9).** — *The equivariant Grayson tower for $Y \times_k K$ is the same with the slice tower of $K^{K,G}$ over Y where $Y \in \mathbf{Sm}/k$ is a semi-local smooth scheme of finite type over k.*

As a consequence, the Grayson spectral sequence is equivalent to the Levine-Serpé spectral sequence for $Y \times_k K$, where Y is a semi-local smooth scheme of finite type over k. In particular, the two spectral sequences are equivalent for G-schemes of dimension 0. When the group G is trivial, our result implies that all the ordinary motivic spectral sequences are equivalent, therefore, recovers the works of Levine, Suslin, Voevodsky, Garkusha-Panin mentioned above.

We should emphasize that this result is the best one we can hope for in general. The two spectral sequences do not need to be equivalent for a semi-local smooth G-scheme of finite type over a field. The typical example is $X = \mathbb{A}^1_0$, the localization of the affine line $\mathbb{A}^1 = \mathrm{Spec}(k[t])$ at the origin and $G = \mathbb{Z}/2$ acts on X by mapping $t \to -t$. In this case, $CH^0(G, X, 0) \cong K_0(k(t^2)) = \mathbb{Z}$ and $H^0_{Gr}(G, X, 0) \cong K_0(G, X) \cong \mathbb{Z} \oplus \mathbb{Z}$. The reason behind this phenomenon is that, the Gersten resolution does *not* exist for equivariant K-theory. In the non-equivariant setting, the Zariski sheaf associated to the presheaf K_0 is the locally constant sheaf $\mathbb{Z}$, i.e., the Grothendieck group of a (semi-)local smooth scheme is isomorphic to the one of its fraction field which is no longer true in the equivariant setting.

06. — Recently, many efforts are putting into constructing and understanding the equivariant version of motivic homotopy theory, for instance, Voevodsky [**Del09**], Hu-Kriz-Ormsby [**HKO10**], Carlsson-Joshua[**CJ11**], Herrmann [**Her13**], Heller-Krishna-Østvær [**HKO15**], Hoyois [**Hoy16**], etc. In some papers, the authors introduced the notations of *equivariant motivic cohomology of Bredon type* which satisfy certain expected properties. In some of these constructions, the equivariant algebraic K-theory is representable. However, we do not know how equivariant K-theory relates to the corresponding equivariant cohomologies represented in these equivariant motivic homotopy categories. The general theory for motivic Postnikov tower in the equivariant setting is still under investigation. Our work is hoped to give some insights into this problem.

Organization of the paper

Chapter 1: Background. — In this chapter, we will collect some general properties of the Morel-Voevodsky's category of S^1-spectra $\mathcal{SH}_{S^1}(k)$ and the Voevodsky's category of effective motives $DM^{eff}(k)$ which are natural frameworks for our study. After that, we will discuss about the slice tower for S^1-spectra and effective motives. In the remaining part, we will recall the construction of the Levine's homotopy coniveau tower for spectra and motives together with the comparison between slice and homotopy coniveau tower.

Chapter 2: Equivariant motivic spectral sequences. — The first two sections are used to recall the constructions of the equivariant Grayson's and Levine-Serpé's spectral sequences. In the last section, we make some remark about the equivariant higher Chow groups appeared in Levine-Serpé's paper, following the spirit of [**Vis91**].

Chapter 3: Equivariant Grayson cohomology. — In this chapter, we discuss about Grayson cohomology: homotopy invariance, ring structure and cancellation. We show that the natural map

$$H^p_{Gr}(G, X, q) \xrightarrow{\sim} H^{p+1}_{Gr}(G, X, q+1)$$

is an isomorphism for all $p, q \in \mathbb{Z}$.

Chapter 4: Comparison of cohomology theories. — Here we consider a special type of action on varieties. Let K be a field with action of a finite group G. Denote by $k := K^G$ the subfield of K which is fixed by the operation of G. Assume that the order of G is coprime to the exponential characteristic of K and the field K is perfect, then so k is. We study the two spectral sequences in case the variety has the form $X \times_k K$ where $X \in \mathbf{Sm}/k$ and G acts on $X \times_k K$ via its action on K. We show that the equivariant Grayson cohomology groups and the equivariant higher Chow groups of $X \times_k K$ are isomorphic (after re-indexing) when X is a semi-local smooth scheme of finite type over k. More precisely, we show that the equivariant Grayson and cycle complexes are slices of certain complexes of Nisnevich sheaves on the category of smooth schemes over k.

Chapter 5: Comparison of spectral sequences. — Using the assumptions and results in Chapter 4, we show the entries and maps in Levine-Serpé's and Grayson's towers are the same, by comparing them to the slice tower of $K^{K,G}$. As a consequence, the differential maps in the two spectral sequences coincide.

Notations and Conventions

Throughout this paper, we will fix a base field k and a finite group G, where G acts trivially on k. The category of separated noetherian schemes of finite type over k is denoted by $\mathbf{Sch}/k$. We use the notations $\mathbf{Sm}/k$ for the full subcategory of smooth schemes in $\mathbf{Sch}/k$ and $G\mathbf{Sm}/k$ for the category of $X \in \mathbf{Sm}/k$ together with a left G-action and morphisms are G-equivariant morphisms in $\mathbf{Sm}/k$. If x is a point on a scheme, its residue field is denoted by $k(x)$.

We let $\mathbf{Ab}$ to be the category of abelian groups and $\mathbf{Spc}_\bullet$ the category of pointed simplicial sets. The stable homotopy category of spectra will be denoted by $\mathcal{SH}$. If $\mathcal{A}$ is an abelian category, the derived category of $\mathcal{A}$ will be denoted by $D(\mathcal{A})$.

For a given a simplicial set, or more general, a simplicial category $S_\bullet$, its geometric realization will be denoted by $|S_\bullet|$. If X is a set, we also use the notation $|X|$ for its cardinality.

We use the notations $\mathbb{A}^1$ for the affine line, $\mathbb{P}^1$ for the projective line and $\mathbb{G}_m$ for the multiplicative group. The circle is denoted by S^1.

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CHAPTER 1

BACKGROUND

1.1. The motivic Postikov tower

1.1.1. The Postnikov tower for S^1-spectra. — The motivic homotopy category of S^1-*spectra* was defined by Jardine in [**Jar00**] following works of Morel-Voevodsky in [**MV99**]. This is a generalization of the classical stable homotopy category in topology. In this section, we will briefly recall its construction.

Let $\mathbf{Spc}_\bullet(k)$ be the category of pointed presheaves of simplicial sets on $\mathbf{Sm}/k$. There is a model structure called *Nisnevich- and* $\mathbb{A}^1$*-local model structure* (or *motivic model structure*) on $\mathbf{Spc}_\bullet(k)$ (cf. [**MV99**]). Denote by $\mathcal{H}_\bullet(k)$ the homotopy category of $\mathbf{Spc}_\bullet(k)$ with respect to this model structure. $\mathcal{H}_\bullet(k)$ is called the *unstable motivic homotopy category* on $\mathbf{Sm}/k$.

$\mathbf{Spc}_\bullet(k)$ contains the category of simplicial sets $\mathbf{Spc}_\bullet$ by taking the constant presheaves. The suspension operation

$$\Sigma_s : \mathbf{Spc}_\bullet(k) \to \mathbf{Spc}_\bullet(k)$$

is defined by $\Sigma_s X := X \wedge S^1$.

For $S \in \mathbf{Spc}_\bullet(k)$, let $\pi_n^{\mathbb{A}^1}(S)$ (or $\pi_n(S)$ for simplicity) be the Nisnevich sheaf associated to the presheaf

$$U \mapsto \mathrm{Hom}_{\mathcal{H}_\bullet(k)}(\Sigma_s^n h_U, S)$$

where h_U is the pointed representable presheaf defined by

$$h_U(X) := \mathrm{Hom}_{\mathbf{Sm}/k}(X, U)_+.$$

For convenience, we will sometimes write U_+ instead of h_U.

In the motivic model structure on $\mathbf{Spc}_\bullet(k)$, the cofibrations are generated by the maps of the form

$$h_X \wedge \partial\Delta[n] \to h_X \wedge \Delta[n], \quad X \in \mathbf{Sm}/k,\ n = 0, 1, \dots,$$

and weak equivalences are maps inducing an isomorphism on π_n for all n.

We denote $\mathbf{Spt}_{S^1}(k)$ to be the category of S^1-spectra in $\mathbf{Spc}_\bullet(k)$, i.e., the category whose objects are sequences $(E_0, E_1, \dots)$ in $\mathbf{Spc}_\bullet(k)$ together with bonding maps $b_n : \Sigma_s E_n \to E_{n+1}$; morphisms are sequences of morphisms in $\mathbf{Spc}_\bullet(k)$ commuting with the bonding maps. Hence, $\mathbf{Spt}_{S^1}(k)$ is just the category of presheaves of classical spectra on $\mathbf{Sm}/k$.

For $E = (E_0, E_1, \ldots) \in \mathbf{Spt}_{S^1}(k)$, the stable homotopy sheaf is defined by

$$\pi_n^s(E) := \lim_{m\to\infty} \pi_{n+m} E_m.$$

A map between presheaves of spectra is *stable weak equivalent* if the induced map on the stable homotopy groups is an isomorphism.

There is a model structure on $\mathbf{Spt}_{S^1}(k)$ defined by Hovey in [**Hov01**] that is called the *stable model structure*, where the weak equivalences are the stable weak equivalences. The homotopy category of $\mathbf{Spt}_{S^1}(k)$ with respect to the stable model structure is denoted by $\mathcal{SH}_{S^1}(k)$.

The infinite suspension functor

$$\begin{aligned} \Sigma_s^\infty : \mathbf{Spc}_\bullet(k) &\to \mathbf{Spt}_{S^1}(k) \\ X &\mapsto (X, \Sigma_s X, \Sigma_s^2 X, \ldots) \end{aligned}$$

and the 0-space functor

$$\begin{aligned} \Omega_s^\infty : \mathbf{Spt}_{S^1}(k) &\to \mathbf{Spc}_\bullet(k) \\ (E_0, E_1, \ldots) &\mapsto E_0 \end{aligned}$$

form a Quillen pair $(\Sigma_s^\infty, \Omega_s^\infty)$. This pair induces a pair of adjoint functors on the homotopy categories

$$\Sigma_s^\infty : \mathcal{H}_\bullet(k) \rightleftharpoons \mathcal{SH}_{S^1}(k) : \Omega_s^\infty.$$

Let $\mathbb{G}_m$ be the pointed space $(\mathbb{A}^1\backslash\{0\}, 1)$, we set $T := S^1 \wedge \mathbb{G}_m$. Denote by Σ_T the operation $- \wedge T$ on $\mathbf{Spt}_{S^1}(k)$. This operation admits a right adjoint the T-loops functor $\Omega_T(-) := \mathcal{H}om(T, -)$.

Consider $\mathbb{P}^1$ as a pointed space using ∞ as the base point. In $\mathcal{SH}_{S^1}(k)$ we have an isomorphism $\mathbb{P}^1 \cong S^1 \wedge \mathbb{G}_m$. Denote by $\Sigma_{\mathbb{P}^1} := - \wedge \mathbb{P}^1$, then $\Sigma_{\mathbb{P}^1} \cong \Sigma_T$.

Consider the localizing subcategory $\Sigma^d_{\mathbb{P}^1}\mathcal{SH}_{S^1}(k)$ of $\mathcal{SH}_{S^1}(k)$ generated by the $\mathbb{P}^1$-suspension $\Sigma^d_{\mathbb{P}^1} E$ for $E \in \mathcal{SH}_{S^1}(k)$. This forms the tower of localizing subcategories

$$\Sigma^n_{\mathbb{P}^1}\mathcal{SH}_{S^1}(k) \subset \Sigma^{n-1}_{\mathbb{P}^1}\mathcal{SH}_{S^1}(k) \subset \ldots \subset \Sigma^0_{\mathbb{P}^1}\mathcal{SH}_{S^1}(k) = \mathcal{SH}_{S^1}(k).$$

The category $\mathcal{SH}_{S^1}(k)$ is compactly generated. It is generated by compact objects $\Sigma^\infty X_+$ where we identify X with the presheaf of simplicial sets represented by X. By Neeman's Brown representability theorem for compactly generated triangulated category, the inclusion

$$i_n : \Sigma^n_{\mathbb{P}^1}\mathcal{SH}_{S^1}(k) \to \mathcal{SH}_{S^1}(k)$$

admits a right adjoint $r_n : \mathcal{SH}_{S^1}(k) \to \Sigma^n_{\mathbb{P}^1}\mathcal{SH}_{S^1}(k)$.

Voevodsky defined $f_n := i_n \circ r_n : \mathcal{SH}_{S^1}(k) \to \mathcal{SH}_{S^1}(k)$. For any $E \in \mathcal{SH}_{S^1}(k)$, this yields the natural tower

$$\ldots \to f_n E \to f_{n-1} E \to \ldots \to f_0 E = E \tag{2}$$

called the *motivic Postnikov tower* or the *slice tower* for S^1-spectra.

The slice tower satisfies the following universal property: For any $F \in \Sigma^n_{\mathbb{P}^1}\mathcal{SH}_{S^1}(k)$ and a morphism $\phi : F \to E$, there exists unique morphism $\phi_n : F \to f_n E$ such that the following diagram

$$\begin{tikzcd} & f_nE \arrow[dr] & \\ F \arrow[ur, dotted, "\phi_n"] \arrow[rr, "\phi"] & & E \end{tikzcd}$$

commutes.

For $E \in \mathcal{SH}_{S^1}(k)$, we write $f_{n/n+r}E$ the cofiber of $f_{n+r}E \to f_nE$. We denote by $s_n := f_{n/n+1}$ the nth slice in the Postnikov tower.

1.1.2. The Postnikov tower for motives. — The category of *effective motives* $DM^{eff}(k)$, defined by Voevodsky in [**Voe00a**], is a generalization of the homotopy category of complexes of abelian groups $D(\mathbf{Ab})$.

Denote by $Cor(\mathbf{Sm}/k)$ the category of finite correspondences over k whose objects are objects in $\mathbf{Sm}/k$ and morphisms from X to Y are given by the finite correspondences $Cor(X, Y)$, the group of cycles on $X \times Y$ generated by integral closed subschemes $W \subset X \times Y$ such that $W \to X$ is finite and surjective over some components of X. Composition in $Cor(\mathbf{Sm/k})$ is given by the usual formula

$$W' \circ W := p_{X,Z*}(p^*_{X,Y}(W) \bullet p^*_{Y,Z}(W'))$$

where $W \in Cor(X, Y)$, $W' \in Cor(Y, Z)$ and " $\bullet$" is the intersection product for algebraic cycles (cf. [**MVW06**, Chapter 1]). Sending $f : X \to Y$ to its graph $\Gamma_f \subset X \times Y$ defines a functor $\mathbf{Sm}/k \to Cor(\mathbf{Sm/k})$.

Definition 1.1. — *A presheaf with transfers is an additive contravariant functor $F : Cor(\mathbf{Sm/k}) \to \mathbf{Ab}$.*

The category of presheaves with transfers over k is denoted by $PST(k)$. It is obvious that a presheaf with transfers is a presheaf on $\mathbf{Sm}/k$.

Let $C(PST(k))$ be the category of unbounded complexes of presheaves with transfers on $\mathbf{Sm}/k$. Again, there is a model structure on $C(PST(k))$ called *motivic model structure* (cf. [**KL10**, Appendix C]). The category of effective motives $DM^{eff}(k)$ is the homotopy category of $C(PST(k))$ with respect to this model structure. $DM^{eff}(k)$ is canonically identified with the full triangulated subcategory of $\mathbb{A}^1-$*local objects* in $D(\mathrm{Sh}^{tr}_{Nis}(k))$ the derived category of Nisnevich sheaves with transfers on $\mathbf{Sm}/k$. Recall that $F \in D(\mathrm{Sh}^{tr}_{Nis}(k))$ is $\mathbb{A}^1-$local if for every $X \in \mathbf{Sm}/k$ and $n \in \mathbb{Z}$, the canonical map

$$\mathrm{Hom}_{D(\mathrm{Sh}^{tr}_{Nis}(k))}(X, F[n]) \to \mathrm{Hom}_{D(\mathrm{Sh}^{tr}_{Nis}(k))}(X \times \mathbb{A}^1, F[n])$$

is an isomorphism. Equivalently, F is $\mathbb{A}^1-$local if the presheaf $X \mapsto H^n_{Nis}(X, F)$ is homotopy invariant for every $X \in \mathbf{Sm}/k$ and $n \in \mathbb{Z}$.

Proposition 1.2. — [**KL10**, Theorem C.3.2] *$DM^{eff}(k)$ is a triangulated tensor category with internal Hom functors.*

For any $X \in \mathbf{Sm}/k$, denote by $\mathbb{Z}(X)$ the presheaf on $\mathbf{Sm}/k$ represented by X. Its image in $DM^{eff}(k)$ is denoted by $M(X)$ and called the motive of X. Under the tensor structure, $M(X) \otimes M(Y) \cong M(X \times Y)$ in $DM^{eff}(k)$.

Denote by $DM^{eff}(k)(n)[2n]$ the localizing subcategory of $DM^{eff}(k)$ generated by objects $M(X)(n)[2n]$, $\in \mathbf{Sm}/k$. They form a tower of localizing subcategories

$$\ldots \subset DM^{eff}(k)(n+1) \subset DM^{eff}(k)(n) \subset \ldots \subset DM^{eff}(k)(0) = DM^{eff}(k).$$

By Neeman's Brown representability theorem, the inclusion

$$i_n^{mot} : DM^{eff}(k)(n) \to DM^{eff}(k)$$

admits a right adjoint r_n^{mot}. Let $f_n^{mot} := i_n^{mot} \circ r_n^{mot}$ then for any $E \in \mathcal{SH}_{S^1}(k)$, we have the *motivic Postnikov tower* or the *slice tower* of E in $DM^{eff}(k)$

$$\ldots \to f_{n+1}^{mot}E \to f_n^{mot}E \to \ldots \to f_0^{mot}E = E. \tag{3}$$

As in $\mathcal{SH}_{S^1}(k)$, the nth slice $s_n^{mot}E$ is defined by the cone of $f_{n+1}^{mot}E \to f_n^{mot}E$ in $DM^{eff}(k)$. The motivic Postnikov tower plays the same role as the tower given by truncation functor in the derived category of abelian groups $D(\mathbf{Ab})$.

There is a *motivic Eilenberg-MacLane funtor* $\mathrm{EM}_{\mathbb{A}^1} : DM^{eff}(k) \to \mathcal{SH}_{S^1}(k)$, defined in [**DRØ03**] (see also [**KL10**]. This uses the same construction of the classical Eilenberg-MacLane in topology. This functor admits a left adjoint Mot, such that $(\mathrm{Mot}, \mathrm{EM}_{\mathbb{A}^1})$ is a Quillen pair. Moreover,

Proposition 1.3. — ([**KL10**, Propostion 1.4.4]) *There is canonical isomorphism for all $n \geq 0$,*

$$\mathrm{EM}_{\mathbb{A}^1} \circ f_n^{mot} \cong f_n \circ \mathrm{EM}_{\mathbb{A}^1}, \quad \mathrm{EM}_{\mathbb{A}^1} \circ s_n^{mot} \cong s_n \circ \mathrm{EM}_{\mathbb{A}^1}.$$

In other words, Postnikov towers are preserved under these functors.

1.2. The homotopy coniveau tower

1.2.1. General construction. — In order to generalize the motivic spectral sequences for algebraic K-theory to larger class of spectra, Levine introduced the notion of *homotopy coniveau tower* in [**Lev08**]. This is an analogue of the Gersten resolution in algebraic K-theory. It turns out to be a very useful construction. Firstly, it makes very clear which formal properties of algebraic K-theory needed to produce motivic spectral sequence, therefore recovers the results of Friedlander-Suslin [**FS02**] and Levine [**Lev01**]. Secondly, it gives a nice description of the slice tower, in terms of algebraic cycles. In this section, we will introduce some notations and mention some properties of the theory which will be used in the next chapters.

Let $\Delta^\bullet$ be the cosimplicial scheme defined by

$$\Delta^r := \mathrm{Spec}\big(k[t_0, \ldots, t_r]/(\Sigma t_i = 1)\big).$$

The *vertices* of Δ^r are the closed subschemes v_i^r defined by $t_i = 1$ and $t_j = 0$ for $j \neq i$. A *face* of Δ^r is a closed subscheme defined by $t_{i_1} = \ldots = t_{i_j} = 0$ for $\{i_1, \ldots, i_j\} \subset \{0, \ldots, r\}$.

For $X \in \mathbf{Sm}/k$, let $S_X^{(p)}(r)$ denote the set of closed subsets $W \subset X \times \Delta^r$ in good position with respect to faces, i.e.,

$$\mathrm{codim}_{X \times F}(W \cap X \times F) \geq p$$

for all faces F of Δ^r. Sending r to $S_X^{(p)}(r)$ defines a simplicial set $S_X^{(p)}(-)$. Let $X^{(p)}(r)$ be the set of codimension p points x of $X \times \Delta^r$ with closure $\bar{x} \in S_X^{(p)}(r)$.

Let $E \in \mathbf{Spt}_{\bullet}(k)$ be a presheaf of spectra. For $X \in \mathbf{Sm}/k$ with closed subscheme W and open complement $j : X\backslash W \to X$, we denote $E^W(X)$ to be the homotopy fiber of $j^* : E(X) \to E(X\backslash W)$.

Let $E^{(p)}(X, r)$ be the filtered homotopy colimit

$$E^{(p)}(X, r) := \mathrm{hocolim}_{W \in S_X^{(p)}(r)} E^W(X \times \Delta^r).$$

Sending r to $E^{(p)}(X, r)$ defines a simplicial spectrum $E^{(p)}(X, -)$. Clearly, $S_X^{(p+1)}(r)$ is a subset of $S_X^{(p)}(r)$, we have a tower of simplicial spectra

$$\ldots \to E^{(p+1)}(X, -) \to E^{(p)}(X, -) \to \ldots \to E^{(0)}(X, -) \tag{4}$$

which is called the *homotopy coniveau tower* of $E(X)$. The cofiber of the map $E^{(p+1)}(X, -) \to E^{(p)}(X, -)$ is denoted by $E^{(p/p+1)}(X, -)$.

For any $F \in C(PST(k))$ and $X \in \mathbf{Sm}/k$, we make the analogous definition to obtain the simplicial complexes $F^{(p)}(X, -)$.

For any $E \in \mathbf{Spt}_{\bullet}(k)$ and a smooth map $f : Y \to X$ we have a well-defined map $f^* : E^{(p)}(X, r) \to E^{(p)}(Y, r)$ for any r, that extend to a map of simplicial spectra $f^* : E^{(p)}(X, -) \to E^{(p)}(Y, -)$. Moreover, the following diagram

$$\begin{tikzcd}
E^{(p+1)}(X, -) \arrow[r, "f^*"] \arrow[d] & E^{(p+1)}(Y, -) \arrow[d] \\
E^{(p)}(X, -) \arrow[r, "f^*"] & E^{p}(Y, -).
\end{tikzcd}$$

commutes. Hence the homotopy coniveau tower is natural with respect to smooth pullback.

Definition 1.4. — (1) *An elementary Nisnevich square is a Cartesian diagram*

$$\begin{tikzcd}
U \times_X V \arrow[r] \arrow[d] & V \arrow[d, "p"] \\
U \arrow[r, "i"] & X
\end{tikzcd} \tag{5}$$

of smooth schemes over k with p étale, i is an open immersion and $p^{-1}(X\backslash U) \cong X\backslash U$.

(2) *A presheaf of spectra E on $\mathbf{Sm}/k$ is said to have Nisnevich excision property if for all elemetary Nisnevich square* (5)*, the diagram*

$$\begin{tikzcd}
E(X) \arrow[r] \arrow[d] & E(U) \arrow[d, "p"] \\
E(V) \arrow[r, "i"] & E(U \times_X V)
\end{tikzcd}$$

is homotopy Cartesian. In other words, $p^* : E^{W'}(X') \to E^W(X)$ is a weak equivalence where $W = X\backslash U$ and $W' = p^{-1}(W) \cong W$.

Using techniques of the Chow's moving lemma, it is showed that if E is a homotopy invariant and have Nisnevich excision property then the homotopy coniveau tower (4) can be made to be functorial over $\mathbf{Sm}/k$, see [**Lev08**, Theorem 4.1.1]. We denote by $E^{(p)}$ the functor

$$\begin{aligned} E^{(p)} : \mathbf{Sm}/k \to & \mathbf{Spt} \\ X \mapsto & E^{(p)}(X,-). \end{aligned}$$

These functors yield the homotopy coniveau tower of E

$$\ldots \to E^{(p+1)} \to E^{(p)} \to \ldots \to E^{(0)} \cong E. \tag{6}$$

The following theorem, due to Levine, gives a very precise formula of the slice tower

Theorem 1.5. — ([**Lev08**, Theorem 7.1.1]) *Suppose that k is a perfect infinite field. For $E \in \mathcal{SH}_{S^1}(k)$ and $n \geq 0$ an integer, $E^{(n)}$ is in $\Sigma^n_{\mathbb{P}^1}\mathcal{SH}_{S^1}(k)$ and the map $E^{(n)} \to f_nE$ induced by the canonical map $E^{(n)} \to E$ is an isomorphism. In other words, the towers* (2) *and* (6) *are equivalent.*

The proof for $E \in \mathbf{Spt}_{S^1}(k)$ apply to $F \in C(PST(k))$ to get

Theorem 1.6. — *Let k be a infinite perfect field. For any $F \in DM^{eff}(k)$ and $n \in \mathbb{N}$, we have $F^{(n)} \in DM^{eff}(k)(n)$ and the natural map $F^{(n)} \to f^{mot}_n F$ is an isomorphism.*

Remark 1.7. — [**KL10**, Remark 2.2.6] *Let $E \in \mathbf{Spt}_{S^1}(k)$ that satisfies homotopy invariant and Nisnevich excision, Theorem 1.5 gives an explicit description of s_0E. For any field extension F/k, denote by $\hat{\Delta}^\bullet_F$ the cosimplicial scheme of semi-local n-simplices where $\hat{\Delta}^n_F$ is the localization of Δ^n with respect to its vertices. Then for every $X \in \mathbf{Sm}/k$, $(s_0E)(X)$ is weakly equivalent to the total spectrum $E(\hat{\Delta}^\bullet_{k(X)})$.*

1.2.2. Well-connected theory. — For some special objects E in $\mathbf{Spt}_{S^1}(k)$ and $C(PST(k))$ called well-connected theories, the slice s_nE (resp. $s^{mot}_n E$) has a cycle-theoretic description via a generalization of the Bloch's higher Chow group. We will briefly recall some definitions and properties. For more details, see [**Lev08**, Section 5,6] or [**KL10**, Part I.3].

Proposition 1.8. — ([**Lev08**, Corollary 5.3.2]) *Let $E : \mathbf{Sm}/k^{op} \to \mathbf{Spt}_\bullet$ be a presheaf satisfying homotopy invariant and Nisnevich excision, then for any $X \in \mathbf{Sm}/k$ the spectrum $E^{(p/p+1)}(X,-)$ is naturally isomorphic to the total spectrum of the simplicial spectra*

$$n \to \coprod_{x \in X^{(p)}(n)} (\Omega^p_T E)^{(0/1)}(k(x))$$

in $\mathcal{SH}$.

Definition 1.9. — *Let $E \in \mathbf{Spt}_{S^1}(k)$ be a homotopy invariant presheaf satisfying Nisnevich excision. E is called well-connected if the followings hold:*

1. *For $X \in \mathbf{Sm}/k$ and $W \subset X$ a closed subset, $E^W(X)$ is -1-connected.*
2. *For any finitely generated field extension F/k, $\pi_n((\Omega^p_T)^{(0/1)}(F) = 0$ for all $n \neq 0$ and $p \geq 0$.*

By Remark 1.7, E is well-connected if for any finitely generated field extension F/k, $\pi_n(\Omega_T^p E)(\hat{\Delta}_F^\bullet) = 0$ for all $n \neq 0$ and $p \geq 0$.

Let $E \in \mathbf{Spt}_{S^1}(k)$ be a well-connected theory. For $X \in \mathbf{Sm}/k$ and $W \subset X$ a closed subset, let

$$z_W^p(X;E) := \oplus_{x \in X^{(p)}, \bar{x} \subset W} \pi_0(\Omega_T^p E(k(x))).$$

The *higher cycles with E-coefficients* is defined as

$$z^p(X;E,n) := \lim_{W \in S_X^{(p)}(n)} z_W^p(X \times \Delta^n; E).$$

Sending $n \to z^p(X;E,n)$ forms a simplicial abelian group $z^p(X;E,-)$. Denote by $z^p(X;E,*)$ the complex associated to $z^p(X;E,-)$.

Definition 1.10. — *Let $X \in \mathbf{Sm}/k$. The higher Chow groups of X with E-coefficients are the homology groups of the complex $z^p(X;E,*)$, i.e.,*

$$CH^p(X;E,n) := H_n(z^p(X;E,*)).$$

For any well-connected spectrum E and $X \in \mathbf{Sm}/k$, there is a *cycle class map*

$$cl_p(X) : E^{(p/p+1)}(X,-) \to \mathrm{EM}(z^p(X,E,-)) \tag{7}$$

defined as follows: Since E is -1-connected, so are $\Omega_T^p E$ and $(\Omega_T^p E)^{(0/1)}$. We have therefore for every $X \in \mathbf{Sm}/k$ a map

$$f : (\Omega_T^p E)^{(0/1)}(X) \to \mathrm{EM}(\pi_0(\Omega_T^p E)^{(0/1)}(X)). \tag{8}$$

The natural morphism

$$g : \pi_0(\Omega_T^p E)(F) \to \pi_0(\Omega_T^p E)^{(0/1)}(F) \tag{9}$$

is an isomorphism for every finitely generated field extension F/k and every well-connected theory E (see [**Lev08**, Lemma 6.1.3]). The cycle class map cl_p in (7) is the composition of the isomorphism in Proposition 1.8 with the map f in (8) and the map $\mathrm{EM}(g^{-1})$ in (9).

Theorem 1.11. — ([**Lev08**, Section 6])

(1) *Let $E : \mathbf{Sm}/k^{op} \to \mathbf{Spt}_\bullet$ be a well-connected theory, the cycle class map*

$$cl_p(X) : E^{(p/p+1)}(X,-) \to \mathrm{EM}(z^p(X;E,-))$$

is a weak equivalence for each $X \in \mathbf{Sm}/k$.

(2) *Let $F \in C(PST(k))$ be well-connected. The cycle class map*

$$cl_p(X) : F^{(p/p+1)}(X,-) \to z^p(X;F,-)$$

is a weak equivalence for each $X \in \mathbf{Sm}/k$.

Corollary 1.12. — *For a well-connected theory E, there is a strongly convergent spectral sequence*

$$E_1^{p,q} := CH^p(X;E,-p-q) \Rightarrow E_{-p-q}(X). \tag{10}$$

Proof. — The spectral sequence (10) is a general one arising from exact couple (see [**Gra05**, Section 4]). The strong convergence comes from the fact that $\mathrm{EM}(z^p(X;E,-))$ is $(p-d-1)$-connected where $d := \dim X$ (dimension reason). □

1.3. Birational motives

Birational motives are defined by Kahn and Sujatha in [**KS02**] in an attempt to understand unramified cohomology from motivic point of view. Roughly speaking, the category of birational motives is obtained from the Voevodsky's category of effective motive $DM^{eff}(k)$ by killing the Lefschetz motive.

Definition 1.13. — *A motive $F \in DM^{eff}(k)$ is called birational if for every dense open immersion $j : U \to X$ in $\mathbf{Sm}/k$ and every $n \in \mathbb{Z}$, the map*

$$j^* : \mathrm{Hom}_{DM^{eff}(k)}(M(X), F[n]) \to \mathrm{Hom}_{DM^{eff}(k)}(M(U), F[n])$$

is an isomorphism.

If $F \cong H^0(F)$ in $D(\mathrm{Sh}^{tr}_{Nis}(k))$, i.e., F is a sheaf then F is called a birational motivic sheaf.

Proposition 1.14. — ([**KL10**, Lemma 4.1.3]) *Let F be a presheaf with transfers that is birational and homotopy invariant then F is a birational motivic sheaf.*

Proof. — Since F is birational presheaf, for every open immersion $U \hookrightarrow X$ we have $F(X) \xrightarrow{\sim} F(U)$. Therefore, for every elementary Nisnevich square (5), the diagram

$$\begin{array}{ccc} F(X) & \longrightarrow & F(U) \\ \downarrow & & \downarrow{\scriptstyle p} \\ F(V) & \xrightarrow{i} & F(U \times_X V) \end{array} \tag{11}$$

is Cartesian. Since every presheaf of sets which transforms coproducts to products and satisfies (11) is a Nisnevich sheaf, so is F.

Since F is birational Nisnevich sheaf, for any open immersion $U \hookrightarrow V$ in the small Nisnevich site of X, the map $F(V) \to F(U)$ is surjective. Hence F is flasque, therefore the Nisnevich cohomology $H^n_{Nis}(X, F) = 0$ for $n \geq 1$. We have

$$\mathrm{Hom}_{D(\mathrm{Sh}^{tr}_{Nis}(k))}(\mathbb{Z}(X), F[n]) \cong H^n_{Nis}(X, F)$$

which implies that $\mathrm{Hom}_{D(\mathrm{Sh}^{tr}_{Nis}(k))}(\mathbb{Z}(X), F[n]) = 0$ for $n \geq 1$. By assumption, F is homotopy invariant, hence every fibrant replacement of F in $D(\mathrm{Sh}^{tr}_{Nis}(k))$ is motivic fibrant replacement of F in $DM^{eff}(k)$, we have

$$\mathrm{Hom}_{DM^{eff}(k)}(M(X), F[n]) = \mathrm{Hom}_{D(\mathrm{Sh}^{tr}_{Nis}(k))}(\mathbb{Z}(X), F[n]) = \begin{cases} F(X), & \text{if } n = 0 \\ 0, & \text{if } n \geq 1 \end{cases}$$

which completes the proof.

□

1.3.1. The Postnikov tower for birational motives.

— For a motive $F \in DM^{eff}(k)$, there is a canonical map

$$\pi_0 : F \to s^{mot}_0 F$$

given by the composition $F \cong f^{mot}_0 F \to s^{mot}_0 F$.

Proposition 1.15. — ([**KL10**, Theorem 4.2.1]) *For $F \in DM^{eff}(k)$, F is birational if and only if $\pi_0 : F \to s^{mot}_0 F$ is an isomorphism. In particular, $s^{mot}_0 F$ is a birational motive.*

Proof. — There are three main ingredients:

(I) The distinguished triangle

$$f_1^{mot}F \to f_0^{mot}F \xrightarrow{\pi_0} s_0^{mot}F \to f_1^{mot}F[1]$$

in $DM^{eff}(k)$ says that π_0 is an isomorphism if and only if $f_1^{mot}F = 0$.

(II) Let $X \in \mathbf{Sm}/k$ that can be assume to be irreducible with open $U \hookrightarrow X$. Let $Z := X \backslash U$ be the complement with reduced structure, we can assume that Z is smooth of codimension $d \geq 1$. Then there is a distinguished triangle

$$M(U) \to M(X) \to M(Z)(d)[2d] \to M(U)[1].$$

in $DM^{eff}(k)$. Since $d \geq 1$,

$$\mathrm{Hom}_{DM^{eff}(k)}(M(Z)(d)[2d], F) = \mathrm{Hom}_{DM^{eff}(k)}(M(Z)(d)[2d], f_1^{mot}F).$$

(III) Assume that F is fibrant. By Remark 1.7, we have

$$\mathrm{Hom}_{DM^{eff}(k)}(M(X), s_0^{mot}F[n]) \cong H^n(F(\hat{\Delta}^*_{k(X)})).$$

Now, if π_0 is an isomorphism then $f_1^{mot}F[n] = 0$ by (I), hence

$$\mathrm{Hom}_{DM^{eff}(k)}(M(Z)(d)[2d], F[n]) = 0.$$

The long exact sequence induced by (II) implies that

$$\mathrm{Hom}_{DM^{eff}(k)}(M(X), F[n]) \xrightarrow{\sim} \mathrm{Hom}_{DM^{eff}(k)}(M(U), F[n]),$$

i.e., F is birational.

Conversely, if F is birational, by taking the limit over all open subset $U \subset X$, we have

$$\mathrm{Hom}_{DM^{eff}(k)}(M(X), F[n]) \xrightarrow{\sim} \mathrm{Hom}_{DM^{eff}(k)}(M(\mathrm{Spec}(k(X))), F[n]).$$

By (III) we have

$$\mathrm{Hom}_{DM^{eff}(k)}(M(X), s_0^{mot}F[n]) \xrightarrow{\sim} \mathrm{Hom}_{DM^{eff}(k)}(M(\mathrm{Spec}(k(X))), s_0^{mot}F[n]).$$

We only need to show that

$$\mathrm{Hom}_{DM^{eff}(k)}(M(\mathrm{Spec}(k(X))), F[n]) \xrightarrow{\sim} \mathrm{Hom}_{DM^{eff}(k)}(M(\mathrm{Spec}(k(X))), s_0^{mot}F[n]),$$

or equivalently, the map

$$F(\Delta^*_{k(X)}) \to F(\hat{\Delta}^*_{k(X)})$$

is weakly equivalent for fibrant F. Since F is birational, the map $F(\Delta^d_{k(X)}) \to F(\hat{\Delta}^d_{k(X)})$ is a weak equivalence for any $d \in \mathbb{N}$, so is the corresponding map between the two total spectra.

Therefore, the identity

$$\mathrm{Hom}_{DM^{eff}(k)}(M(X), f_1^{mot}F[n]) = 0$$

holds for any $X \in \mathbf{Sm}/k$. Since $DM^{eff}(k)$ is generated by $M(X)$ for $X \in \mathbf{Sm}/k$, it implies that $f_1^{mot}F[n] = 0$ for all $n \in \mathbb{N}$. □

Corollary 1.16. — *Let F be a birational motive. Then*

$$f_m^{mot}(F(n)) = \begin{cases} 0, & \text{for } m > n, \\ F(n), & \text{for } m \leq n. \end{cases}$$

1.3.2. Birational motivic sheaves. —

Lemma 1.17. — ([**KL10**, Proposition 4.3.2]) *If F is a birational motivic sheaf then $F(q)[2q]$ is well-connected for all $q \geq 0$.*

Proposition 1.18. — ([**KL10**, Theorem 4.3.3]) *If F is a birational motivic sheaf then for $q \geq 0$, there is a natural isomorphism*

$$H^{2q-p}(X, F(q)) := \mathrm{Hom}_{DM^{eff}(k)}(M(X), F(q)[2q-p]) \cong \mathrm{CH}^q(X; F(q)[2q], p).$$

Proof. — Since $F(q)[2q]$ is well-connected, $s_q^{mot} F(q)[2q]$ is computed by cycle complex by Theorem 1.11, i.e.,

$$\mathrm{Hom}_{DM^{eff}(k)}(M(X), s_q^{mot}(F(q)[2q])[-p]) \cong CH^q(X, F(q)[2q], q).$$

By Corollary (1.16), $s_q^{mot}(F(q)[2q]) \cong F(q)[2q]$ that concludes the proposition. □

1.4. K_0- and $K_0^{\oplus}$-presheaves

In order to study the Grayson spectral sequence, Walker consider in [**Wal96**] a class of presheaves on $\mathbf{Sm}/k$ called K_0-presheaves which have similar properties to the presheaves with transfers defined in the previous section.

For $X, Y \in \mathbf{Sm}/k$, denote by $\mathcal{P}(X; Y)$ the category of coherent $\mathcal{O}_{X\times Y}$-modules P such that Supp P is finite over X and the coherent $\mathcal{O}_X$-module $(p_X)_*(P)$ is locally free.

Let $K_0^{\oplus}(X, Y) := K_0^{\oplus}(\mathcal{P}(X, Y))$ and $K_0(X, Y) := K_0(\mathcal{P}(X, Y))$. Recall that for a given exact category $\mathcal{C}$, the Grothendick group $K_0(\mathcal{C})$ of $\mathcal{C}$ is the abelian group generated by objects $X \in \mathcal{C}$ modulo the relation $[X] = [X'] + [X'']$ whenever $0 \to X' \to X \to X'' \to 0$ is a short exact sequence in $\mathcal{C}$. The direct sum Grothendieck group $K_0^{\oplus}(\mathcal{C})$ of $\mathcal{C}$ is defined using the same generator as $K_0(\mathcal{C})$ but we only allow split short exact sequences to give relations.

For $X, Y, Y \in \mathbf{Sm}/k$, we have a natural biexact bifunctor

$$\begin{aligned} \circ : \mathcal{P}(X, Y) \otimes \mathcal{P}(Y, Z) &\to \mathcal{P}(X, Z), \\ (P, Q) &\mapsto (p_{X,Z})_*(p_{X,Y}^*(P) \otimes_{\mathcal{O}_{X\times Y\times Z}} p_{Y,Z}^*(Q)) \end{aligned}$$

where $p_{X,Y}$, $p_{Y,Z}$, $p_{X,Z}$ are projections from $X\times Y\times Z$ to $X\times Y$, $Y\times Z$, $X\times Z$, respectively. This gives a natural composition law

$$K_0^{\oplus}(X, Y) \otimes K_0^{\oplus}(Y, Z) \to K_0^{\oplus}(X, Z).$$

The similar statement holds for K_0.

Denote by $K_0^{\oplus}(\mathbf{Sm}/k)$ (resp. $K_0(\mathbf{Sm}/k)$) the category whose objects are $X \in \mathbf{Sm}/k$ and morphisms $\mathrm{Hom}_{K_0^{\oplus}(\mathbf{Sm}/k)}(X, Y) := K_0^{\oplus}(X, Y)$ (resp. $\mathrm{Hom}_{K_0(\mathbf{Sm}/k)}(X, Y) := K_0(X, Y)$) for $X, Y \in \mathbf{Sm}/k$. There is a natural functor

$$\mathbf{Sm}/k \to K_0^{\oplus}(\mathbf{Sm}/k)$$

by sending a morphism $f : X \to Y$ in $\mathbf{Sm}/k$ to the class $\mathcal{O}_{\Gamma_f} \in K_0^{\oplus}(\mathbf{Sm}/k)$ of its graph $\Gamma_f \subset X \times Y$.

There are obvious functors

$$K_0^{\oplus}(\mathbf{Sm}/k) \to K_0(\mathbf{Sm}/k) \to Cor(\mathbf{Sm}/k)$$

where the first functor is given by the natural surjections $K_0^{\oplus}(X,Y) \to K_0(X,Y)$ and the second functor is given by $K_0(X,Y) \to Cor(X,Y)$ mapping every $[F] \in K_0(X,Y)$ to its support.

Definition 1.19. — *A $K_0^{\oplus}$-presheaf (resp. K_0-presheaf) on $\mathbf{Sm}/k$ is a additive contravariant functor $F : K_0^{\oplus}(\mathbf{Sm}/k) \to \mathbf{Ab}$ (resp. $K_0^{\oplus}(\mathbf{Sm}/k) \to \mathbf{Ab}$).*

A K_0-presheaf F is a $K_0^{\oplus}$-presheaf by composing with the natural functor $K_0^{\oplus}(\mathbf{Sm}/k) \to K_0(\mathbf{Sm}/k)$. The following lemma says that, up to homotopy invariance, $K_0^{\oplus}$-presheaves become K_0-presheaves.

Lemma 1.20. — ([**Wal96**]) *Let F be a homotopy invariant $K_0^{\oplus}$-presheaf on $\mathbf{Sm}/k$. Then the $K_0^{\oplus}$-presheaf structure on F descends uniquely to a K_0-presheaf structure.*

Obviously, a $K_0^{\oplus}$-presheaves and K_0-presheaves are presheaves of abelian group on $\mathbf{Sm}/k$. Similar to the statement for presheaves with transfers, we have the following

Lemma 1.21. — ([**Wal96**]) *Let F be a homotopy invariant K_0-presheaf. Then the associated Zariski sheaf F_{Zar} has a unique structure of a K_0-presheaf for which the canonical homomorphism $F \to F_{Zar}$ is a homomorphism of K_0-presheaves. Moreover, F is a homotopy invariant presheaf and has a canonical structure of a homotopy invariant pretheory.*

Homotopy invariant pretheory is defined by Voevodsky in [**Voe00b**]. One of the main features of such theory is that the Zariski sheaves associated to them have the Gersten resolutions and the hypercohomologies of sheaves for Nisnevich and Zariski are the same.

Proposition 1.22. — (Voevodsky, [**Voe00b**]) *Let $F : \mathbf{Sm}/k \to \mathbf{Ab}$ is a homotopy invariant pretheory. Assume that $X \in \mathbf{Sm}/k$ is a semi-local scheme with function field K, then the natural morphism*

$$F(X) \to F(K)$$

is injective.

Proposition 1.23. — (Voevodsky, [**Voe00b**]) *Let F be a homotopy invariant pretheory with values in $\mathbf{Ab}$. Assume that the base field k is perfect. Then the natural homomorphism*

$$H^i_{Nis}(X, F_{Nis}) \to H^i_{Zar}(X, F_{Zar})$$

is an isomorphism for all $X \in \mathbf{Sm}/k$ and $i \geq 0$.

CHAPTER 2

EQUIVARIANT MOTIVIC SPECTRAL SEQUENCES

2.1. Equivariant Grayson tower

In [**Gra95**] Grayson constructed a spectral sequence for algebraic K-theory on regular affine Noetherian schemes, following a suggestion of Goodwillie and Lichtenbaum. The tower that produces this spectral sequence is obtained by considering the K-theory spectrum of of finite generated projective modules together with commuting automorphisms. The construction applies easily to the equivariant setting. In this section, we will recall some facts needed to produce this spectral sequence.

Let $\mathcal{M}$ be an exact category, S a ring. Denote by $\mathcal{M}(S)$ the category of pairs (P, ϕ) where P is an object of $\mathcal{M}$ and $\phi : S \to \mathrm{End}_{\mathcal{M}}(P)$ is an *S-module structure* for P. A morphism $f : (P_1, \phi_1) \to (P_2, \phi_2)$ in $\mathcal{M}(S)$ is a morphism $\bar{f} : P_1 \to P_2$ in $\mathcal{M}$ compatible with ϕ_1 and ϕ_2 in an obvious way. f is a *monomorphism* (resp. *epimorphism*) if $\bar{f}$ is monomorphism (resp. epimorphism). Hence $\mathcal{M}(S)$ is an exact category.

Example 2.1. — (1) *If $S = \mathbb{Z}$ is the ring of integer numbers then $\mathcal{M}(\mathbb{Z}) = \mathcal{M}$.*

(2) *If $S = \mathbb{Z}[U, U^{-1}]$ is the multiplicative group then $\mathcal{M}(\mathbb{Z}[U, U^{-1}])$ is the exact category of pairs (P, ϕ) where P is an object of $\mathcal{M}$ and ϕ is an automorphism of P. We will denote this category by $\mathcal{M}(\mathbb{G}_m)$.*

Let R be a commutative ring with unit 1 equipped with an action of a finite group G, denote by $\mathcal{P}(G, R)$ the *exact category* of finitely generated projective G-equivariant modules on R. For a commutative ring with unit S we set $\mathcal{P}(G, R, S) := \mathcal{P}(G, R)(S)$, then $\mathcal{P}(G, R, S)$ is the exact category of R-S-bimodules which are finitely generated projective G-equivariant modules on R. In the language of schemes, if $X = \mathrm{Spec}(R)$ and $Y = \mathrm{Spec}(S)$ then $\mathcal{P}(G, R, S)$ is equivalent to $\mathcal{P}(G, X, Y)$ the category of G-equivariant modules Q on $X \times Y$ such that $\mathrm{Supp}(Q)$ is finite over X and $(p_X)_*Q$ is a G-vector bundle on X (see section 1.4).

Denote by $K(G, R, \mathbb{G}_m^n)$ and $K^{\oplus}(G, R, \mathbb{G}_m^n)$ the K-theory spectrum and direct sum K-theory spectrum of $\mathcal{P}(G, R, \mathbb{G}_m^n)$, respectively. We identify $K(G, R, \mathbb{G}_m^0)$ with $K(G, R, \mathbb{Z}) = K(G, R)$, the equivariant K-theory spectrum of $\mathcal{P}(G, R)$.

Denote by $K(G, R, \mathbb{G}_m^{\wedge 1}) := \mathrm{hofib}(K(G, R, \mathbb{G}_m^1) \to K(G, R, \mathbb{Z}))$ the homotopy fiber of the natural map $h : K(G, R, \mathbb{G}_m^1) \to K(G, R, \mathbb{Z})$. Since $\mathrm{Spec}\ \mathbb{Z} \to \mathbb{G}_m \to \mathrm{Spec}\ \mathbb{Z}$ is the identity map, the map h splits and we have

$$K(G,R,\mathbb{G}_m^1) = K(G,R) \times K(G,R,\mathbb{G}_m^{\wedge 1}).$$

Iterating this process, we obtain a spectrum $K(G,R,\mathbb{G}_m^{\wedge n})$. Replacing K by $K^{\oplus}$, we obtain $K^{\oplus}(G,R,\mathbb{G}_m^{\wedge n})$.

The n-cube $[n]$ is the category of subsets (including the empty set) of $\{1,\dots,n\}$ with maps being inclusions of subsets. An n-cube in a category $\mathcal{C}$ is a covariant from $[n]$ to $\mathcal{C}$.

Let $\mathcal{M}$ be an exact category, we denote by $\mathcal{M}(\mathbb{G}_m^{\boxtimes n})$ the n-cube of categories given by

$$\mathcal{M}(\mathbb{G}_m^{\boxtimes n})(I) = \mathcal{M}(\mathbb{G}_m^{|I|})$$

with obvious morphisms. Apply the K- and direct sum K-theory spectrum to $\mathcal{M}(\mathbb{G}_m^{|I|})$, we obtain two n-cubes of spectra $K(\mathcal{M}(\mathbb{G}_m^{\boxtimes n}))$ and $K^{\oplus}(\mathcal{M}(\mathbb{G}_m^{\boxtimes n}))$, respectively.

We set

$$K(\mathcal{M}(\mathbb{G}_m^{\wedge n})) := \mathrm{hocofib}(K\mathcal{M}(\mathbb{G}_m^{\boxtimes n}))$$

and

$$K^{\oplus}(\mathcal{M}(\mathbb{G}_m^{\wedge n})) := \mathrm{hocofib}(K^{\oplus}\mathcal{M}(\mathbb{G}_m^{\boxtimes n}))$$

be the iterated homotopy cofibers of the n-cubes of spectra (see [**Gra95**, section 1] for more details).

Recall that a simplicial ring $R_\bullet$ is *contractible* if there exists a homotopy

$$H : \Delta^1 \times R_\bullet \to R_\bullet$$

from 0 to 1. If $R_\bullet$ is a connected simplicial rings $R_\bullet$, i.e., there is an element $T \in R_1$ such that $\delta_0^* T = 0$ and $\delta_1^* T = 1$ where $\delta_0, \delta_1 : R_0 \to R_1$ are face maps then $R_\bullet$ is contractible.

Lemma 2.2. — ([**Gra95**, Corollary 9.6]) *Let $R_\bullet$ be a contractible simplicial ring, $\mathcal{M}_\bullet$ an $R_\bullet-$linear simplicial exact category. Then there is a tower*

$$\dots \longrightarrow W^{n+1} \longrightarrow W^n \longrightarrow \dots \longrightarrow W^0 = |K^{\oplus}(\mathcal{M}_\bullet)|$$

where

$$W^n = \Omega^{-n}|d \to K^{\oplus}\mathcal{M}_d(\mathbb{G}_m^{\wedge n})|,$$

and successive homotopy cofibers are of the form

$$W^n/W^{n+1} = \Omega^{-n}|d \to K_0^{\oplus}\mathcal{M}_d(\mathbb{G}_m^{\wedge n})|.$$

Lemma 2.3. — ([**Gra95**, Theorem 10.5]) *Let $R_\bullet$ and $\mathcal{M}_\bullet$ as in Lemma 2.2, the natural map*

$$|K^{\oplus}\mathcal{M}_\bullet| \to |K\mathcal{M}_\bullet|$$

is a homotopy equivalence.

Let $R_\bullet = R\mathbb{A}^\bullet$ be the simplicial ring where

$$R\mathbb{A}^d = R[t_0,\dots,t_d]/(\sum t_i = 1),$$

then $R_\bullet$ is contractible because it is obviously connected. Let $\mathcal{M}_\bullet$ be the $R_\bullet-$linear category with $\mathcal{M}_d = \mathcal{P}(G, R\mathbb{A}^d)$ where G acts trivially on $t_0,\dots,t_d$. It is obvious that

$$K^{\oplus}\mathcal{M}_d(\mathbb{G}_m^{\wedge n}) = K^{\oplus}(G, R\mathbb{A}^d, \mathbb{G}_m^{\wedge n})$$

and

$$K_0^{\oplus}\mathcal{M}_d(\mathbb{G}_m^{\wedge n}) = K_0^{\oplus}(G, R\mathbb{A}^d, \mathbb{G}_m^{\wedge n}).$$

For each $i \in \mathbb{N}$, consider

$$\mathcal{M}_d^i := \mathcal{P}(G, R\mathbb{A}^d, \mathbb{G}_m^i).$$

By Lemma 2.3, we have

$$|K^{\oplus}(G, R\mathbb{A}^\bullet, \mathbb{G}_m^i)| \sim |K(G, R\mathbb{A}^\bullet, \mathbb{G}_m^i)|$$

Taking the iterated homotopy cofibers with respect to the n-cubes $K^{\oplus}(G, R\mathbb{A}^\bullet, \mathbb{G}_m^{\boxtimes n})$ and $K(G, R\mathbb{A}^\bullet, \mathbb{G}_m^{\boxtimes n})$, we get

$$|K^{\oplus}(G, R\mathbb{A}^\bullet, \mathbb{G}_m^{\wedge n})| \sim |K(G, R\mathbb{A}^\bullet, \mathbb{G}_m^{\wedge n})|$$

In particular,

$$|K^{\oplus}(G, R\mathbb{A}^\bullet)| \sim |K(G, R\mathbb{A}^\bullet)|.$$

If R is regular then $K(G, R) \sim |K(G, R\mathbb{A}^\bullet)|$ by homotopy invariant property. Combining all the above data we have

Theorem 2.4. — *Let R be a regular noetherian ring. Then there is a tower*

$$\ldots \longrightarrow W^{n+1}(G, R) \longrightarrow W^n(G, R) \longrightarrow \ldots \longrightarrow W^0(G, R) \sim K(G, R) \tag{12}$$

where

$$W^n(G, R) = \Omega^{-n}|K(G, R\mathbb{A}^\bullet, \mathbb{G}_m^{\wedge n})|,$$

and successive homotopy cofibers are of the form

$$W^n(G, R)/W^{n+1}(G, R) \sim \Omega^{-n}|K_0^{\oplus}(G, R\mathbb{A}^\bullet, \mathbb{G}_m^{\wedge n})|.$$

The tower (12) *is called the equivariant Grayson tower.*

For any G-scheme X, denote by $K_0^{\oplus}(G, X \times \Delta^*, \mathbb{G}^{\wedge n})$ the complex associated to the simplicial abelian group

$$r \to K_0^{\oplus}(G, X \times \Delta^r, \mathbb{G}^{\wedge n}).$$

Definition 2.5. — *The Grayson complex $C^*_{Gr}(G, X, n)$ is defined by*

$$C^*_{Gr}(G, X, n) := K_0^{\oplus}(G, X \times \Delta^{-*}, \mathbb{G}^{\wedge n})[-n], \tag{13}$$

with the obvious differential maps.

The equivariant Grayson cohomology groups are defined by

$$H^p_{Gr}(G, X, n) := H^p(C^*_{Gr}(G, X, n)). \tag{14}$$

Corollary 2.6. — *For regular affine smooth G-scheme X, there is a strongly convergent spectral sequence*

$$E_2^{p,q} := H^{p-q}_{Gr}(G, X, -q) \Rightarrow K_{-p-q}(G, X), \tag{15}$$

that is called the equivariant Grayson spectral sequence.

Proof. — As in Lemma 1.12, there is a strongly convergent spectral sequence

$$E_1^{p,q} := \pi_{-p-q}(\Omega^{-p}|K_0^{\oplus}(G, X \times \Delta^\bullet, \mathbb{G}_m^{\wedge p})|) \Rightarrow K_{-p-q}(G, X).$$

We have

$$\begin{aligned} E_1^{p,q} &:= \pi_{-p-q}(\Omega^{-p}|K_0^{\oplus}(G, X \times \Delta^\bullet, \mathbb{G}_m^{\wedge p})|) \\ &= \pi_{-2p-q}|K_0^{\oplus}(G, X \times \Delta^\bullet, \mathbb{G}_m^{\wedge p})| \\ &= H^{2p+q}(K_0^{\oplus}(G, X \times \Delta^{-*}, \mathbb{G}^{\wedge p})) \\ &= H^{3p+q}(C^*_{Gr}(G, X, p)) \\ &= H^{3p+q}_{Gr}(G, X, p), \end{aligned} \tag{16}$$

with differential $r_1^{p,q} : E_1^{p,q} \to E_1^{p+1,q}$. By changing $p \to -q$ and $3p+q \to p-q$, we obtain $r_2^{p,q} : E_2^{p,q} \to E_2^{p+2,q-1}$, hence a strongly convergent spectral sequence

$$E_2^{p,q} := H^{p-q}_{Gr}(G, X, -q) \Rightarrow K_{-p-q}(G, X).$$

□

Remark 2.7. — 1. *The statement of Theorem 2.4 is not only true for finite groups but also for algebraic groups, whenever they satisfy the conditions* [**Tho83**, Corollary 5.8]. *In this case, we have a homotopy equivalence*

$$K(G, X) \xrightarrow{\sim} G(G, X),$$

hence homotopy invariance for $K(G, X)$.

2. *The Grayson spectral sequence is contravariantly functorial for morphisms between regular affine Noetherian G-schemes. It is obvious from its construction.*
3. *There is a multiplication on equivariant Grayson cohomology induced from the product on $K_*(G, X)$ (cf. section 3.2).*

Remark 2.8. — *Grayson's construction is common known as the simplest construction of the motivic spectral sequence. Unfortunately, the cohomology groups appeared in unexpected forms. The intrusion of the direct-sum Grothendieck groups in the construction was unwelcome. M. Walker later made extensive progress on this approach and then showed that the fibers of the Grayson's tower give the correct theory of motivic cohomology in weights smaller than two* [**Wal01**]. *Using Walker's results on K_0-presheaves, Suslin later showed that the Grayson cohomology is the same with the one defined by Voevodsky over semi-local smooth schemes of finite type over a field (cf.*[**Sus03**]*), that is generally believed to be the right candidate for motivic cohomology.*

2.2. The Levine-Serpé tower

In order to produce a motivic spectral sequence for equivariant algebraic K-theory, Levine and Serpé modified the idea of the homotopy coniveau tower in Section 1.2. In [**LS08**], they showed that the E^2-terms in the spectral sequence have cycle-theoretic description via a generalization of the Bloch's higher Chow group.

For $X \in G\mathbf{Sm}/k$, let

$$S^{(p)}_{G,X}(r) := \left\{ W \subset X \times \Delta^r | \begin{array}{c} W \text{ is a closed } G-\text{stable subset} \\ \text{and for all faces } F \subset \Delta^r \text{ we have} \\ \mathrm{codim}_{X\times F}(W \cap X \times F) \geq p \end{array} \right\}.$$

Sending r to $S^{(p)}_{G,X}(r)$ defines a simplicial set.

We set

$$K^{(p)}(G, X, r) := \mathrm{hocolim}_{W \in S^{(p)}_{G,X}(r)} K^W(G, X \times \Delta^r).$$

Then the sequence of subsets

$$\ldots \subset S^{(p+1)}_{G,X}(r) \subset S^{(p)}_{G,X}(r) \subset \ldots$$

gives the *homotopy coniveau tower* for equivariant K-theory

$$\ldots \to K^{(p+1)}(G, X, -) \to K^{(p)}(G, X, -) \to \ldots \to K^{(0)}(G, X, -).$$

Denote by

$$K^{(p/p+1)}(G, X, -) := \mathrm{hocofib}(K^{(p+1)}(G, X, -) \to K^{(p)}(G, X, -))$$

the pth layer of the homotopy coniveau tower.

The K-theory spectrum is homotopy invariant on $G\mathbf{Sm}/k$, we have $K^{(0)}(G, X, -) = K(G, X, -) \sim K(G, X)$.

Let

$$X^{(p)}_G(r) := \{[x] \in (X \times \Delta^r)^{(p)}/G | \ \overline{G.x} \in S^{(p)}_{G,X}(r)\}$$

Define $z^p(G, X, r)$ by

$$z^p(G, X, r) := \bigoplus_{[x] \in X^{(p)}_G(r)} K_0(G_x, k(x))$$

where $G_x \subset G$ is the isotropy group for x. Sending r to $z^p(G, X, r)$ defines a simplicial abelian group. The *equivariant cycle complex of Bredon type* $z^p(G, X, *)$ is by definition the complex of abelian groups associated to $z^p(G, X, -)$.

Definition 2.9. — ([**LS08**, Definition 3.4]) *The equivariant higher Chow groups of Bredon type are defined by*

$$CH^p(G, X, r) := \pi_r(z^p(G, X, -)) = H_r(z^p(G, X, *)).$$

(compare to Definition 1.10).

There is a *cycle map*

$$cl_p : K^{(p/p+1)}(G, X, -) \to z^p(G, X, -) \tag{17}$$

defined as follows:

The canonical homomorphism

$$\pi_0(K^{(p)}(G, X \times \Delta^r)) \to \bigoplus_{[x] \in X^{(p)}_G(r)} K_0(G_x, k(x))$$

factors through the surjection

$$\pi_0(K^{(p)}(G, X \times \Delta^r)) \to \pi_0(K^{(p/p+1)}(G, X \times \Delta^r)).$$

Consider $\pi_0(K^{(p/p+1)}(G, X \times \Delta^r))$ and $z^p(G, X \times \Delta^r)$ as spectra (Eilenberg-MacLane spectrum). Since $K^{(p/p+1)}(G, X \times \Delta^r)$ is -1-connected for each n, we have a map of simplicial spectra

$$K^{(p/p+1)}(G, X, \Delta^r) \to \pi_0(K^{(p/p+1)}(G, X, \Delta^r))$$

and hence a map

$$K^{(p/p+1)}(G, X \times \Delta^r) \to z^p(G, X \times \Delta^r)$$

which yield a map of simplicial spectra

$$cl_p : K^{(p/p+1)}(G, X, -) \to z^p(G, X, -).$$

Using localization techniques developed in [**Lev01**] to reduces to the case of points and handles the case of points by the techniques of the homotopy coniveau machinery in Section 1.2, Levine and Serpé proved the following

Theorem 2.10. — ([**LS08**, Theorem 3.7]) *Let $X \in G\mathbf{Sm}/k$ and suppose that $\frac{1}{|G|} \in k$, then the cycle map*

$$cl_p : K^{(p/p+1)}(G, X, -) \to z^p(G, X, -)$$

is a weak equivalence for all p.

As a consequence, we have

Corollary 2.11. — *For $X \in G\mathbf{Sm}/k$ and suppose that $(|G|, \mathrm{char} k) = 1$. There is a strongly convergent spectral sequence*

$$E_1^{p,q} = CH^p(X, -p-q) \Rightarrow K_{-p-q}(G, X). \tag{18}$$

Proof. — See Lemma 1.12. □

Remark 2.12. — 1. *When we index the equivariant higher Chow groups by dimension rather than codimension, we use the subscript $CH_p(G, X, q)$. In this case, there is a strongly convergent spectral sequence*

$$E_1^{p,q} := CH_{-p}(G, X, -p-q) \Rightarrow G_{-p-q}(G, X) \tag{19}$$

without assuming X is smooth [**LS08**, Corollary 3.8]. *The equivariant higher Chow groups $CH_p(G, X, q)$ satisfy localization property and the spectral sequence* (19) *is compatible with localization* [**LS08**, Theorem 4.1].

2. *The Levine-Serpé's construction is contravariantly functorial in X for flat equi-dimensional maps in $G\mathbf{Sm}/k$ and contravariantly functorial in G. It is covariantly functorial for proper maps in $G\mathbf{Sm}/k$.*
3. *There is no way to make the Levine-Serpé's tower functorial for arbitrary morphisms in $G\mathbf{Sm}/k$. This contradicts to the ordinary case, when we can use the Bloch's moving cycles to make the homotopy coniveau tower contravariantly functorial in* $\mathbf{Sm}/k$ [**LS08**, Remark 3.6].
4. *The equivariant higher Chow groups are not homotopy invariant in general. For example, let $G = \mathbb{Z}/n$ acts on $\mathbb{A}^1_k$ by multiplication the nth roots of unity, we have*

$$CH^1(G, \mathbb{A}^1_k, 0) \cong K_0(k[G])/(reg) \cong \mathbb{Z}^{n-1}.$$

Here reg is the regular representation of $\mathbb{Z}/n$ over k [**LS08**, Example 6.17]. *On the other hand, $CH^1(G, \mathrm{Spec} k, 0) = 0$ by dimension reason. Therefore, pull-back* *by the flat G-equivariant projection $p : \mathbb{A}^1_k \to \operatorname{Spec} k$ does not give an isomorphism $p^* : CH^1(G, \operatorname{Spec} k, 0) \to CH^1(G, \mathbb{A}^1_k, 0)$, even after tensoring with $\mathbb{Q}$.*

However, $CH^p(G, -, q)$ do satisfy homotopy invariant with respect to the projection $X \times \mathbb{A}^1_k \to X$, if G acts on $X \times \mathbb{A}^1_k$ via the given action on X and the trivial action on $\mathbb{A}^1_k$ [**KS02**, Corollary 5.6].

5. *There is no obvious ring structure on $CH^*(G, X, *)$. This come from the fact that the topological filtration on $K(G, X)$ is not functorial with respect to pull-back. Therefore, we cannot expect the product on $K_*(G, X)$ to induce a product on $CH^*(G, X, *)$ in the usual way.*

2.3. Some discussions

If $H \subset G$ is a subgroup, we can define the induced map

$$\operatorname{Ind}_H^G : CH^p(H, X, q) \to CH^p(G, X, q)$$

as follows: If $[x] \in (X \times \Delta^r)_{(p+r)}/H$ such that the closure $\overline{H.x} \in S^{H,X}_{(p+r)}(r)$ is in good position, then $\overline{G.x} \in S^{G,X}_{(p+r)}(r)$ is also in good position. Each orbit $H.x$ gives rise to an orbit $G.x$. It is obvious that the isotropy group H_x is a subgroup of G_x. Therefore, we have the induced map for equivariant K-theory

$$\operatorname{Ind}_{H_x}^{G_x} : K_0(H_x, k(x)) \to K_0(G_x, k(x)) \tag{20}$$

(see [**Vis91**, Section 2]). Taking the sum over all orbits $[x] \in (X \times \Delta^r)_{(p+r)}/H$, we obtain the induced map for cycle complexes, hence the desired map for equivariant higher Chow groups.

Let X be a projective scheme over a field k of characteristic p and G be a finite group of order n acting on X. Assume that $(n, p) = 1$ and k contains n-th roots of unity. Let Γ be the set of representatives for the conjugacy classes of cyclic subgroups of G. For each $\sigma \in \Gamma$ denote by $N(\sigma)$ the normalizer of σ in G and X^σ the fixed point scheme of σ. Since the order of σ is relatively prime to the characteristic of k, the closed subscheme X^σ is regular if X is.

If $\sigma \in \Gamma$ we denote by $R\sigma$ the representation ring of σ over k. Let t be the generator of the dual group $\hat{\sigma}$ of homomorphisms $\sigma \to k^*$ and m be the order of σ, then we have

$$R\sigma \cong \mathbb{Z}\hat{\sigma} \cong \mathbb{Z}[t]/(t^m - 1) \cong \prod_{d|m} \mathbb{Z}[t]/(\Phi_d(t))$$

where $\Phi_d(t)$ is the d-th cyclotomic polynomial. Denote by $\tilde{R}\sigma$ the factor of $R\sigma$ corresponding to $\mathbb{Z}[t]/(\Phi_m(t))$ which is independent of the choice of the generator t. The group $N(\sigma)$ acts on X^σ and therefore it acts on $K_*(X^\sigma)$. It also acts on σ by conjugation, hence it acts on $\tilde{R}\sigma$.

Proposition 2.13. — ([**Vis91**, Theorem 1 and 2])

(1) *If X is regular then there exists a canonical isomorphism of graded $\mathbb{Z}$-algebras*

$$K_*(G, X) \cong \prod_{\sigma \in \Gamma} (K_*(X^\sigma) \otimes \tilde{R}\sigma)^{N(\sigma)}. \tag{21}$$

(2) *There exists a canonical isomorphism of graded $\mathbb{Z}$-algebras*

$$G_*(G, X) \cong \prod_{\sigma \in \Gamma} (G_*(X^\sigma) \otimes \tilde{R}\sigma)^{N(\sigma)} \tag{22}$$

which is compatible with localization sequence.

The isomorphisms (21) and (22) are vast generalizations of results on the ring of representations of finite groups over a field presented in [**Ser77**]. They are motivated by a result of R. Segal on the comparison between equivariant (topological) K-theory of a compact oriented differentiable manifold and the equivariant singular cohomology of its fixed point sets (cf. [**HH90**]).

In the ordinary setting, there is a strongly convergent spectral sequence [**Lev01**, Proposition 8.9]

$$E_1^{p,q} := CH_{-p}(X, -p-q) \Rightarrow G_{-p-q}(X). \tag{23}$$

When G acts trivially on X, we have an obvious isomorphism

$$CH_p(X, q) \otimes RG \xrightarrow{\sim} CH_p(G, X, q). \tag{24}$$

We construct the morphism

$$\pi : \prod_{\sigma \in \Gamma} (CH_p(X^\sigma, q) \otimes \tilde{R}\sigma)^{N(\sigma)} \to CH_p(G, X, q) \tag{25}$$

as the composition of the following maps: the inclusion

$$\prod_{\sigma \in \Gamma} (CH_p(X^\sigma, q) \otimes \tilde{R}\sigma)^{N(\sigma)} \to \prod_{\sigma \in \Gamma} (CH_p(X^\sigma, q) \otimes R\sigma)^{N(\sigma)},$$

the isomorphism

$$\prod_{\sigma \in \Gamma} (CH_p(X^\sigma, q) \otimes R\sigma)^{N(\sigma)} \to \prod_{\sigma \in \Gamma} CH_p(\sigma, X^\sigma, q)^{N(\sigma)},$$

the inclusion

$$\prod_{\sigma \in \Gamma} CH_p(\sigma, X^\sigma, q)^{N(\sigma)} \to \prod_{\sigma \in \Gamma} CH_p(\sigma, X^\sigma, q),$$

the push-forward

$$\prod_{\sigma \in \Gamma} CH_p(\sigma, X^\sigma, q) \to \prod_{\sigma \in \Gamma} CH_p(\sigma, X, q),$$

and the induced map

$$\prod_{\sigma \in \Gamma} CH_p(\sigma, X, q) \to CH_p(G, X, q).$$

The map π is presumed compatible with two spectral sequences (19) and (23). It is reasonable to expect that these spectral sequence compatible with localization sequence. Therefore we expect the morphism π is an isomorphism. We will consider this problem in another paper.

However, when we index the higher Chow groups by codimension , we do not have an isomorphism similar to (25). For instance, let $G := \mathbb{Z}/2$ acts on $X := \mathbb{A}^1$ by sending t to $-t$. By simple calculation, we see that

$$CH^1(\mathbb{Z}/2, \mathbb{A}^1, 0) = \mathbb{Z}$$

(c.f [**LS08**, Example 6.17]).

The group $\mathbb{Z}/2$ has only 2 cyclic subgroups 0 and $\mathbb{Z}/2$. We obtain

$$CH^1((\mathbb{A}^1)^0, 0) = CH^1(\mathbb{A}^1, 0) = CH^1(\mathrm{Spec} k, 0) = 0$$

and

$$CH^1((\mathbb{A}^1)^{\mathbb{Z}/2}, 0) = CH^1(\mathrm{Spec} k, 0) = 0$$

by dimension reason. Therefore

$$\prod_{\sigma \in \Gamma} (CH^1((\mathbb{A}^1)^\sigma, 0) \otimes \tilde{R}\sigma)^{N(\sigma)} = 0.$$

CHAPTER 3

EQUIVARIANT GRAYSON COHOMOLOGY

3.1. Homotopy invariance

Lemma 3.1. — *Let $\mathbb{A}^1$ be the affine line with trivial action of G, the natural map*

$$H^p_{Gr}(G, X, q) \to H^p_{Gr}(G, X \times \mathbb{A}^1, q) \tag{26}$$

is an isomorphism.

Proof. — Let $i_\alpha : X \to X \times \mathbb{A}^1$ be the inclusion $x \mapsto (x, \alpha)$, denote by

$$i^*_\alpha : H^p_{Gr}(G, X \times \mathbb{A}^1, q) \to H^p_{Gr}(G, X, q)$$

the pull-back map. Then (26) holds if and only if $i^*_0 = i^*_1$.

Now we use the standard simplicial decomposition of the polyhedron $\Delta^n \times \Delta^1$. For any $i = 0, \dots, n$, let $\theta_i : \Delta^{n+1} \to \Delta^n \times \mathbb{A}^1$ be the map that sends the vertex v_j to $v_j \times \{0\}$ for $j \le i$ and to $v_{j-1} \times \{1\}$ otherwise. For any G-scheme X, we have the induced maps

$$(1_X \times \theta_i)^* : K^{\oplus}_0(G, X \times \mathbb{A}^1 \times \Delta^n, \mathbb{G}^{\wedge q}) \to K^{\oplus}_0(G, X \times \Delta^{n+1}, \mathbb{G}^{\wedge q}).$$

Let $h_n := \Sigma(-1)^i(1_X \times \theta_i)^*$ then h_n is a chain homotopy from i^*_1 to i^*_0 on the level of complex. Taking pth - homology groups we have $i^*_0 = i^*_1$. □

Lemma 3.2. — *If $F : G\mathbf{Sm}/k \to \mathbf{Ab}$ is a contravariant functor such that $F(X) \to F(X \times \mathbb{A}^1)$ is an isomorphism for any $X \in G\mathbf{Sm}/k$ and G acts trivially on $\mathbb{A}^1$ then for any representation V of G over k the natural morphism*

$$F(X) \to F(X \times \mathbb{A}(V))$$

is an isomorphism, where $\mathbb{A}(V)$ is the affine G-spaces associated to V.

Proof. — The equivariant G-morphism

$$\mu : \mathbb{A}(V) \times \mathbb{A}^1 \to \mathbb{A}(V),$$
$$(x, t) \mapsto t.x,$$

is an equivariant homotopy between the identity on $\mathbb{A}(V)$ and $\mathbb{A}(V) \to \{0\} \subset \mathbb{A}(V)$. □

Corollary 3.3. — *For any representation V of G over k, the natural morphism*

$$H^p_{Gr}(G, X, q) \to H^p_{Gr}(G, X \times \mathbb{A}(V), q)$$

is an isomorphism.

Corollary 3.4. — *For any affine scheme $X \in G\mathbf{Sm}/k$ and any representation V of G over k, the Grayson towers*

$$\ldots \longrightarrow W^2(G,X) \longrightarrow W^1(G,X) \longrightarrow W^0(G,X)$$

and

$$\ldots \longrightarrow W^2(G,X\times\mathbb{A}(V)) \longrightarrow W^1(G,X\times\mathbb{A}(V)) \longrightarrow W^0(G,X\times\mathbb{A}(V))$$

are equivalent.

3.2. Products on Grayson cohomology

For X, Y, Z in $G\mathbf{Sm}/k$, we have the natural bifunctor given by

$$\begin{aligned}\circ : \mathcal{P}(G,X,Y)\otimes\mathcal{P}(G,Y,Z) &\to \mathcal{P}(G,X,Z),\\ (F,\mathcal{G}) &\mapsto F\circ\mathcal{G}\end{aligned}$$

where

$$F\circ\mathcal{G} = (p_{X,Z})_*(p^*_{X,Y}(F)\otimes_{\mathcal{O}_{X\times Y\times Z}} p^*_{Y,Z}(\mathcal{G})).$$

It is not hard to see that this functor is well-defined and biexact. Therefore, we have a natural composition law

$$K_0^{\oplus}(G,X,Y)\otimes K_0^{\oplus}(G,Y,Z)\to K_0^{\oplus}(G,X,Z).$$

Let $X, X', Y, Y' \in G\mathbf{Sm}/k$. For $F\in\mathcal{P}(G,X,Y)$ and $\mathcal{G}\in\mathcal{P}(G,X',Y')$, we have the external tensor product $F\boxtimes\mathcal{G}$ which is obvious finite and flat over $X\times X'$. Therefore we get a bifunctor

$$\mathcal{P}(G,X,Y)\times\mathcal{P}(G,X',Y')\to\mathcal{P}(G,X\times X',Y\times Y')$$

which is clearly additive and biexact. This gives a canonical operation

$$\begin{aligned}\boxtimes : K_0^{\oplus}(G,X,Y)\otimes K_0^{\oplus}(G,X',Y') &\to K_0^{\oplus}(G,X\times X',Y\times Y')\\ ([F],[\mathcal{G}]) &\mapsto [F\boxtimes\mathcal{G}],\end{aligned}$$

and hence

$$K_0^{\oplus}(G,X\times\Delta^*,\mathbb{G}^{\wedge m})\otimes K_0^{\oplus}(G,Y\times\Delta^*,\mathbb{G}^{\wedge n}) \xrightarrow{\boxtimes} \mathrm{Tot}(K_0^{\oplus}(G,X\times Y\times\Delta^*\times\Delta^*,\mathbb{G}^{\wedge m+n})$$

By composing $\boxtimes$ with the shuffle map

$$\mathrm{Tot}(K_0^{\oplus}(G,X\times Y\times\Delta^*\times\Delta^*,\mathbb{G}^{\wedge m+n})\to(K_0^{\oplus}(G,X\times\times\Delta^*,\mathbb{G}^{\wedge m+n})$$

we obtain an operation

$$K_0^{\oplus}(G,X\times\Delta^*,\mathbb{G}^{\wedge m})\otimes K_0^{\oplus}(G,Y\times\Delta^*,\mathbb{G}^{\wedge n}) \xrightarrow{\gamma} K_0^{\oplus}(G,X\times Y\times\Delta^*,\mathbb{G}^{\wedge m+n}).$$

When $X = Y$, the diagonal map $\delta_X : X\to X\times X$ defines an operation

$$\delta_X^* : K_0^{\oplus}(G,X\times X\times\Delta^*,\mathbb{G}^{\wedge m+n})\to K_0^{\oplus}(G,X\times\Delta^*,\mathbb{G}^{\wedge m+n}).$$

The composition

$$\delta_X^*\circ\gamma : K_0^{\oplus}(G,X\times\Delta^*,\mathbb{G}^{\wedge m})\otimes K_0^{\oplus}(G,X\times\Delta^*,\mathbb{G}^{\wedge n})\to K_0^{\oplus}(G,X\times\Delta^*,\mathbb{G}^{\wedge m+n})$$

defines a product

$$H^p_{Gr}(G,X,m)\times H^q_{Gr}(G,X,n)\to H^{p+q}_{Gr}(G,X,m+n)$$

which yields a ring structure on equivariant Grayson cohomology. With this product, the
.

3.3. Cancellation Theorem

Cancellation for motivic cohomology plays a central role in the theory of motives due to Voevodsky (cf. [**Voe10**]). A analogous statement for Grayson cohomology was proved by Suslin in [**Sus03**]. In this section, we will verify this property for equivariant Grayson cohomology.

Let F be a presheaf of abelian groups on $(G)\mathbf{Sm}/k$, we define

$$F(X\wedge\mathbb{G}_m):=\mathrm{Ker}(F(X\times\mathbb{G}_m)\to F(X\times\{1\})=F(X)).$$

In other words, if we consider $F(X)$ as a direct summand of $F(X\times\mathbb{G}_m)$, then $F(X\wedge\mathbb{G}_m)$ is the complementary direct summand of $F(X)$ in $F(X\times\mathbb{G}_m)$.

Theorem 3.5. — *Let $X,Y\in G\mathbf{Sm}/k$ be affine smooth G-schemes and G acts trivially on Y. The natural homomorphism*

$$\mu: K_0^{\oplus}(G,X\times\Delta^*,Y)\to K_0^{\oplus}(G,X\times\Delta^*\wedge\mathbb{G}_m,Y\wedge\mathbb{G}_m),$$
$$[F]\to[F\boxtimes(1_{\mathbb{G}_m}-e)]$$

is an quasi-isomorphism, where $1_{\mathbb{G}_m}:\mathbb{G}_m\to\mathbb{G}_m$ is the identity map and $e=e_{\mathbb{G}_m}:\mathbb{G}_m\to\mathbb{G}$ is the constant map, maps $\mathbb{G}_m$ to its identity, both are considered as elements in $K_0^{\oplus}(\mathbb{G}_m,\mathbb{G}_m)$.

Idea of the proof: We want to construct a map of complexes

$$\rho: K_0^{\oplus}(G,X\times\Delta^*\wedge\mathbb{G}_m,Y\wedge\mathbb{G}_m)\to K_0^{\oplus}(G,X\times\Delta^*,Y)$$

to have the following diagram

$$\begin{tikzcd}
K_0^{\oplus}(G,X\times\Delta^*\wedge\mathbb{G}_m,Y\wedge\mathbb{G}_m) \arrow[r, "\rho"] \arrow[d, shift right, "\mu_1"'] \arrow[d, shift left, "\mu_2"] & K_0^{\oplus}(G,X\times\Delta^*,Y) \arrow[d, "\mu"] \\
K_0^{\oplus}(G,X\times\Delta^*\wedge\mathbb{G}_m\wedge\mathbb{G}_m,Y\wedge\mathbb{G}_m\wedge\mathbb{G}_m) \arrow[r, "\rho^{(2)}"] & K_0^{\oplus}(G,X\times\Delta^*\wedge\mathbb{G}_m,Y\wedge\mathbb{G}_m).
\end{tikzcd}$$

such that:

(1) $\rho\circ\mu\sim 1_{k_0^{\oplus}(G,X\times\Delta^*,Y)}$.
(2) $\mu_1\sim\mu_2$.
(3) $\mu\circ\rho\sim\rho^{(2)}\circ\mu_1$
(4) $\rho^{(2)}\circ\mu_2\sim 1_{K_0^{\oplus}(G,X\times\Delta^*\wedge\mathbb{G}_m,Y\wedge\mathbb{G}_m)}$.

The notion $\rho^{(2)}$ means the map obtained by applying ρ to the second factors $\mathbb{G}_m$.

Unfortunately, the map we expect does not exist (for a reason we will see below), but we still have a way to construct a similar map, where the domain of "ρ" is $L(G,X\times\Delta^*\times\mathbb{G}_m,Y\times\mathbb{G}_m):=\mathbb{Z}(\mathcal{P}(G,X\times\Delta^*\times\mathbb{G}_m,Y\times\mathbb{G}_m))$, the free abelian group generated by $\mathcal{P}(G,X\times\Delta^*\times\mathbb{G}_m,Y\times\mathbb{G}_m)$, such that the four properties mentioned above are satisfied.

Lemma 3.6. — *μ is well-defined*

Proof. — Let $[F] \in K_0^{\oplus}(G, X \times \Delta^* \wedge \mathbb{G}_m, Y \wedge \mathbb{G}_m)$ then $[F]$ is an element in $K_0^{\oplus}(G, X \times \Delta^* \times \mathbb{G}_m, Y \times \mathbb{G}_m)$ satisfying

$$[F] \circ [1_{X\times\Delta^*} \times e] = [1_Y \times e] \circ [F] = 0.$$

For any $[Q] \in K_0^{\oplus}(G, X \times \Delta^*, Y)$, we have

$$[Q \boxtimes (1_{\mathbb{G}_m} - e)] \circ [1_{X\times\Delta^*} \times e] = [Q \boxtimes (e - e^2)] = 0,$$

$$[1_Y \times e] \circ [Q \boxtimes (1_{\mathbb{G}_m} - e)] = [Q \boxtimes (e - e^2)] = 0.$$

□

Step 1: Construct the map "ρ"

Let $A = F[X]$, $B = F[Y]$ be the coordinate rings of X and Y, respectly. Let $\mathcal{P}$ be a G-coherent sheaf on $X \times \mathbb{G}_m \times Y \times \mathbb{G}_m$ that is finite and flat over $X \times \mathbb{G}_m$. Then $\mathcal{P}$ is a G-module on $A[f_1, f_1^{-1}] \otimes B[f_2, f_2^{-1}]$ that is finitely generated projective G-module on $A[f_1, f_1^{-1}]$.

Lemma 3.7. — ([**Sus03**, Proposition 4.1]).

1. *For any $n \geq 0$, the sheaf $\mathcal{P}/(f_1^{n+1} - 1)\mathcal{P}$ is finite and flat over X.*
2. *There exists $N \geq 0$ such that for any $n \geq N$, the sheaf $\mathcal{P}/(f_1^{n+1} - f_2)\mathcal{P}$ is finite and flat over X.*

Definition 3.8. — *We define $\rho_n^+(\mathcal{P}) := \mathcal{P}/(f_1^{n+1}-1)\mathcal{P}$. We say that $\rho_n^-(\mathcal{P}) := \mathcal{P}/(f_1^{n+1}-f_2)\mathcal{P}$ is defined when $\mathcal{P}/(f_1^{n+1} - f_2)\mathcal{P}$ is finite and flat over X.*

When ρ_n^- is defined, we define $\rho_n(\mathcal{P}) := [\rho_n^+(\mathcal{P})] - [\rho_n^-(\mathcal{P})] \in K_0^{\oplus}(G, X, Y)$.

By Lemma 3.7, there exist $N \geq 0$ such that for any $n \geq N$, ρ_n^-, ρ_n are defined.

By definition, for any $\mathcal{P}_1, \mathcal{P}_2$ are G-coherent sheaves on $X \times \mathbb{G}_m \times Y \times \mathbb{G}_m$, those are finite and flat over $X \times \mathbb{G}_m$, we have the following:

i. $\rho_n^+(\mathcal{P}_1 \oplus \mathcal{P}_2) = \rho_n^+(\mathcal{P}_1) \oplus \rho_n^+(\mathcal{P}_2)$.
ii. $\rho_n^-(\mathcal{P}_1 \oplus \mathcal{P}_2) = \rho_n^-(\mathcal{P}_1) \oplus \rho_n^-(\mathcal{P}_2)$ when they are all defined.

Therefore, $\rho_n(\mathcal{P}_1 \oplus \mathcal{P}_2) = [\rho_n(\mathcal{P}_1)] + [\rho_n(\mathcal{P}_2)] \in K_0^{\oplus}(G, X, Y)$.

We see that we do not actually have a well-defined map ρ on $K_0^{\oplus}(G, X \times \mathbb{G}_m, Y \times \mathbb{G}_m)$. Indeed, for any class $[\mathcal{P}]$ in $K_0^{\oplus}(G, X \times \mathbb{G}_m, Y \times \mathbb{G}_m)$, we might not have a common number N such that $\rho_N(\mathcal{Q})$ is defined for every $\mathcal{Q} \in [\mathcal{P}]$.

Let $V \in L(G, X \times \Delta^* \times \mathbb{G}_m, Y \times \mathbb{G}_m)$, we say that ρ_n is defined on V if it is defined on every P appearing in V with non-zero coefficients.

Corollary 3.9. — *For any V in $L(G, X \times \Delta^* \times \mathbb{G}_m, Y \times \mathbb{G}_m)$ that is 0 in $K_0^{\oplus}(G, X \times \Delta^* \times \mathbb{G}_m, Y \times \mathbb{G}_m)$, there exist N such that $\rho_n(V) = 0$ for all $n \geq N$.*

Proof. — Given $M, N \in \mathrm{Obj}(\mathcal{M})$, two classes $[M]$ and $[N]$ in $K_0^{\oplus}(\mathcal{M})$ are the same iff there exist $P \in \mathrm{Obj}(\mathcal{M})$ such that $M \oplus P = N \oplus P$. Using the formula above we get the desired property. □

Step 2: μ is injective.

Lemma 3.10. — ([**Sus03**, Lemma 4.3]) Let Q be a G-coherent sheaf on $X \times Y$ that is finite and flat over X.

(1) $\rho_n^+(Q \boxtimes 1_{\mathbb{G}_m}) \simeq Q^{n+1}$ as G-sheaves, for all $n \geq 0$.

$\rho_n^-(Q \boxtimes 1_{\mathbb{G}_m})$ is defined for all $n \geq 0$ and we have $\rho_n^-(Q \boxtimes 1_{\mathbb{G}_m}) \simeq Q^n$ as G-sheaves on $X \times Y$.

As a consequence we have $\rho_n(Q \boxtimes 1_{\mathbb{G}_m}) = [Q] \in K_0^{\oplus}(G, X, Y)$ that does not depend on n.

(2) $\rho_n^-(Q \boxtimes e)$ is defined for all $n \geq 0$, hence

$\rho_n^+(Q \boxtimes e) \simeq Q^{n+1}$, $\rho_n^-(Q \boxtimes e) \simeq Q^{n+1}$ as G-sheaves.

As a consequence we have $\rho_n(Q \boxtimes e) = 0 \in K_0^{\oplus}(G, X, Y)$ that does not depend on n.

Proof. — We will use the same notation as in the Step 1. If Q is a G-module on $A \otimes B$ then

$$Q \boxtimes 1_{\mathbb{G}_m} \simeq Q[f_1, f_1^{-1}, f_2, f_2^{-1}]/(f_1 - f_2).$$

It implies that

$$Q \boxtimes 1_{\mathbb{G}_m}/(f_1^{n+1} - 1) \simeq Q[f_1, f_1^{-1}, f_2, f_2^{-1}]/(f_1^{n+1} - 1, f_1 - f_2) \simeq Q^{n+1},$$

$$Q \boxtimes 1_{\mathbb{G}_m}/(f_1^{n+1} - f_2) \simeq Q[f_1, f_1^{-1}, f_2, f_2^{-1}]/(f_1 - f_2, f_1^{n+1} - f_2) \simeq Q^n$$

which are compatible with G-action.

Similarly, we have

$$Q \boxtimes e \simeq Q[f_1, f_1^{-1}, f_2, f_2^{-1}]/(f_2 - 1)$$

$$(Q \boxtimes e)/(f_1^{n+1} - 1) \simeq Q[f_1, f_1^{-1}, f_2, f_2^{-1}]/(f_1^{n+1} - 1, f_2 - 1) \simeq Q^{n+1}.$$

$$(Q \boxtimes e)/(f_1^{n+1} - f_2) \simeq Q[f_1, f_1^{-1}, f_2, f_2^{-1}]/(f_1^{n+1} - f_2, f_2 - 1) \simeq Q^{n+1}.$$

□

Proposition 3.11. — *μ is injective.*

Proof. — Let Q be an element in $\mathcal{P}(G, X, Y)$ such that $[Q] \boxtimes [1_{\mathbb{G}_m} - e] = 0 \in K_0^{\oplus}(G, X \times \Delta^* \wedge \mathbb{G}_m, Y \wedge \mathbb{G}_m)$. By Corollary 3.9, there exist $N \gg 0$ such that $\rho_N(Q \boxtimes (1_{\mathbb{G}_m} - e)) = 0 \in K_0^{\oplus}(G, X, Y)$. By Lemma 3.10, $\rho_N(Q \boxtimes (1_{\mathbb{G}_m} - e)) = [Q]$ that finishes the proof. □

Step 3: $\mu_1 \sim \mu_2$, where μ_i is defined as μ applying to the i-th coordinates $\mathbb{G}_m$.

More precisely,

$$\mu_2 := \mu : K_0^{\oplus}(G, X \times \Delta^* \wedge \mathbb{G}_m, Y \wedge \mathbb{G}_m) \to K_0^{\oplus}(G, (X \times \Delta^* \wedge \mathbb{G}_m) \wedge \mathbb{G}_m, (Y \wedge \mathbb{G}_m) \wedge \mathbb{G}_m).$$

$\mu_1 = \phi \circ \mu_2$, where,

$$\phi : K_0^{\oplus}(G, X \times \Delta^* \wedge \mathbb{G}_m \wedge \mathbb{G}_m, Y \wedge \mathbb{G}_m \wedge \mathbb{G}_m) \to K_0^{\oplus}(G, X \times \Delta^* \wedge \mathbb{G}_m \wedge \mathbb{G}_m, Y \wedge \mathbb{G}_m \wedge \mathbb{G}_m).$$

maps $[F]$ to $[1_Y \times \sigma] \circ [F] \circ [1_{X \times} \Delta^* \times \sigma]$, obtained from F by permuting two copies of $\mathbb{G}_m$ both in $X \times \Delta^* \times \mathbb{G}_m \times \mathbb{G}_m$ and $Y \times \mathbb{G}_m \times \mathbb{G}_m$. Here we denote by $\sigma : \mathbb{G}_m \times \mathbb{G}_m \to \mathbb{G}_m \times \mathbb{G}_m$ the permutation of coordinate morphism.

We will prove that $\phi \sim 1_{K_0^{\oplus}(G,X\times\Delta^*\wedge\mathbb{G}_m\wedge\mathbb{G}_m,Y\wedge\mathbb{G}_m\wedge\mathbb{G}_m)}$, by constructing a homotopy between them.

Lemma 3.12. — ([**Sus03**, Proposition 4.6]). There is a coherent sheaf $\mathcal{H}$ on $(\mathbb{G}_m \times \mathbb{G}_m \times \mathbb{A}^1) \times (\mathbb{G}_m \times \mathbb{G}_m)$ finite and flat over $\mathbb{G}_m \times \mathbb{G}_m \times \mathbb{A}^1$ such that:

$$[\mathcal{H}_0] - [\mathcal{H}_1] = [\sigma] - [1_{\mathbb{G}_m} \times i] + [(a,e)] - [(b,e)] - [(e,a)] - [(e,b)] + 2[(e,e)]$$

where $i : \mathbb{G}_m \to \mathbb{G}_m,\ x \to x^{-1}$, a and b are the first and the second coordinate functions on $\mathbb{G}_m \times \mathbb{G}_m$

For $[F] \in K_0^{\oplus}(G, X \times \Delta^* \times \mathbb{G}_m \times \mathbb{G}_m, Y \times \mathbb{G}_m \times \mathbb{G}_m)$, consider the homotopy

$$\Phi_F := [1_Y \boxtimes \mathcal{H}] \circ \big(([F] \circ [1_{X\times\Delta^*\times\mathbb{G}_m} \times i]) \boxtimes 1_{\mathbb{A}^1}\big) + [1_Y \times \sigma] \circ [F] \circ [1_{X\times\Delta^*} \boxtimes \mathcal{H}]$$

in $K_0^{\oplus}(G, X \times \Delta^* \times \mathbb{G}_m \times \mathbb{G}_m \times \mathbb{A}^1, Y \times \mathbb{G}_m \times \mathbb{G}_m)$.

Lemma 3.13. — *Assume that* $[F] \in K_0^{\oplus}(G, X \times \Delta^* \wedge \mathbb{G}_m \wedge \mathbb{G}_m, Y \wedge \mathbb{G}_m \wedge \mathbb{G}_m)$ *then*

$$[\Phi_F|_0] - [\Phi_F|_1] = [\phi(F)] - [F].$$

Proof. — Let $[F] \in K_0^{\oplus}(G, X\times\Delta^*\wedge\mathbb{G}_m\wedge\mathbb{G}_m, Y\wedge\mathbb{G}_m\wedge\mathbb{G}_m)$, for any $\alpha, \beta, \gamma, \eta : \mathbb{G}_m \to \mathbb{G}_m$, we have

$$[1_Y \times (e,\alpha)] \circ [F] = [1_Y \times (\beta, e)] \circ [F] = 0.$$

$$[F] \circ [1_{X\times\Delta^*} \times (e,\gamma)] = [F] \circ [1_{X\times\Delta^*} \times (\eta, e)] = 0.$$

By Lemma 3.12,

$$\begin{aligned}
[\Phi_F|_0] - [\Phi_F|_1] &= [1_Y \times \sigma] \circ [F] \circ [1_{X\times\Delta^*} \times \sigma] - [1_{Y\times\mathbb{G}_m} \times i] \circ [F] \circ [1_{X\times\Delta^*\times\mathbb{G}_m} \times i] \\
&= [1_Y \times \sigma] \circ [F] \circ [1_{X\times\Delta^*} \times \sigma] - [F] \\
&= [\phi(F)] - [F].
\end{aligned}$$

□

The Lemma implies that $\phi \sim 1_{K_0^{\oplus}(G,X\times\Delta^*\wedge\mathbb{G}_m\wedge\mathbb{G}_m,Y\wedge\mathbb{G}_m\wedge\mathbb{G}_m)}$. Therefore $\mu_1 \sim \mu_2$.

Step 4: μ is surjective.

Let $\rho_n^{(2)} : L(G, (X \times \Delta^* \times \mathbb{G}_m) \times \mathbb{G}_m, (Y \times \mathbb{G}_m) \times \mathbb{G}_m) \to K_0^{\oplus}(G, X \times \Delta^* \times \mathbb{G}_m, Y \times \mathbb{G}_m)$ be the map defined by applying ρ_n to the second factors $\mathbb{G}_m$.

For any $Q \in \mathcal{P}(G, X \times \Delta^* \times \mathbb{G}_m, Y \times \mathbb{G}_m)$, we have

$$\mu \circ \rho_n(Q) = [\rho_n(Q) \boxtimes (1_{\mathbb{G}_m} - e)]$$

when $\rho_n(Q)$ is defined.

For any $Q \in \mathcal{P}(G, X \times \Delta^* \times \mathbb{G}_m, Y \times \mathbb{G}_m)$, by direct computation

$$\rho_n^{(2)}((1_Y \times \sigma) \circ (Q \boxtimes 1_{\mathbb{G}_m}) \circ (1_{X\times\Delta^*} \times \sigma)) = [\rho_n(Q) \boxtimes 1_{\mathbb{G}_m}],$$

$$\rho_n^{(2)}((1_Y \times \sigma) \circ (Q \boxtimes e) \circ (1_{X\times\Delta^*} \times \sigma)) = [\rho_n(Q) \boxtimes e],$$

we obtain

$$\rho_n^{(2)}((1_Y \times \sigma) \circ (Q \boxtimes (1_{\mathbb{G}_m} - e) \circ (1_{X\times\Delta^*} \times \sigma)) = [\rho_n(Q) \boxtimes (1_{\mathbb{G}_m} - e)].$$

In other words, we have $\rho_n^{(2)} \circ \mu_1(Q) = \mu \circ \rho_n(Q)$ when they are defined.

Proposition 3.14. — *μ is surjective.*

Proof. — For any $[Q] \in K_0^{\oplus}(G, X \times \Delta^* \times \mathbb{G}_m, Y \times \mathbb{G}_m)$, choose $Q \in L(G, X \times \Delta^* \times \mathbb{G}_m, Y \times \mathbb{G}_m)$ a representative.

By Lemma 3.12, $\mu_1(Q)$ and $\mu_2(Q)$ are in the same class in $K_0^{\oplus}$, where

$$\mu_2([Q]) = [Q \boxtimes (1_{\mathbb{G}_m} - e)]$$
$$\mu_1([Q]) = \phi[Q \boxtimes (1_{\mathbb{G}_m} - e)] = [1_Y \times \sigma] \circ [Q \boxtimes (1_{\mathbb{G}_m} - e)] \circ [1_{X \times \Delta^*} \times \sigma]$$

By Corollary 3.10, there exist $N \gg 0$ such that $\rho_N^{(2)}$ is defined on $((1_Y \times \sigma) \circ (Q \boxtimes (1_{\mathbb{G}_m} - e)) \circ (1_{X \times \Delta^*} \times \sigma) - Q \boxtimes (1_{\mathbb{G}_m} - e))$ and ρ_N is defined on Q. In this case, we have

$$\rho_N^{(2)}((1_Y \times \sigma) \circ (Q \boxtimes (1_{\mathbb{G}_m} - e)) \circ (1_{X \times \Delta^*} \times \sigma)) = \rho_N^{(2)}(Q \boxtimes (1_{\mathbb{G}_m} - e))$$

The left handside is $[\rho_N(Q) \boxtimes (1_{\mathbb{G}_m} - e)]$ and the right handside is $[Q]$ by Lemma 3.10, we obtain the result. $\square$

Combines 4 steps, we obtain the Theorem 3.5.

CHAPTER 4

COMPARISION OF COHOMOLOGY THEORIES

Let K/k be a field extension with an action of a finite group G, where G acts trivially on k. We can assume that $k = K^G$ the fixed field of K under the action of G. We also assume that $(|G|, \mathrm{char}(k)) = 1$. These notations and conventions will be used throughout the rest of this paper.

For a given presheaf of abelian group F on $\mathbf{Sm}/k$, we denote C_*F to be the complex of presheaves where $C_nF(X) := F(X \times \Delta^n)$ and differential maps are given by taking the alternative sums of the restrictions to faces of codimension one. We use the notation C^*F for the same complex but $C^n = C_{-n}$ (cohomological convention).

Consider the functor

$$\begin{aligned} K_0^{\oplus,K,G}(n) : \mathbf{Sm}/k \to & \mathbf{Ab} \\ Y \mapsto & K_0^{\oplus}(G, Y \times_k K, \mathbb{G}_m^{\wedge n}). \end{aligned}$$

It is obvious that $K_0^{\oplus,K,G}(n)$ is a $K_0^{\oplus}$-presheaf (Definition 1.19).

Let

$$\mathbb{Z}_{Gr}^{K,G}(n) := (C^* K_0^{\oplus,K,G}(n))_{Nis}[-n] \tag{27}$$

be the complexes of Nisnevich sheaves associated to $(C^*K_0^{\oplus,K,G}(n))[-n]$. The complex $\mathbb{Z}_{Gr}^{K,G}(n)$ is the global version of the Grayson complex (13) on $\mathbf{Sm}/k$ and similar to the motivic complex $\mathbb{Z}(n)$ defined in [**MVW06**].

Let $\mathbb{Z}^{K,G}$ be the Zariski sheaf associated to the presheaf

$$\begin{aligned} E := K_0^{K,G} : \mathbf{Sm}/k \to & \mathbf{Ab}, \\ X \mapsto & K_0(G, X \times_k K). \end{aligned}$$

The presheaf E is obviously a homotopy invariant K_0-presheaf on $\mathbf{Sm}/k$, hence by Lemma 1.21, $\mathbb{Z}^{K,G}$ is a homotopy invariant K_0-sheaf. If we consider $\mathbb{Z}^{K,G}$ is a complex of sheaves concentrated in degree 0 then the natural morphism

$$\mathbb{Z}^{K,G} \to C^*\mathbb{Z}^{K,G}$$

is a quasi-isomorphism of complexes of Zariski sheaves. We will see later that $\mathbb{Z}^{K,G}$ is actually a sheaf of Nisnevich topology.

Let $\mathbb{Z}^{K,G}(n) := \mathbb{Z}^{K,G} \otimes \mathbb{Z}(n)$ be the tensor product of $\mathbb{Z}^{K,G}$ with the motivic complex $\mathbb{Z}(n)$ in the category of Nisnevich sheaves on $\mathbf{Sm}/k$ (cf. [**MVW06**, Lecture 3]). Since

$\mathcal{Z}(n)$ is a flat complex of sheaves, $\mathbb{Z}^{K,G}(n)$ represents the derived sheaf tensor product $\mathbb{Z}^{K,G} \otimes^L \mathbb{Z}(n)$.

Lemma 4.1. — *There are natural morphisms*

$$\phi_n : \mathbb{Z}^{K,G}(n) \to \mathbb{Z}^{K,G}_{Gr}(n) \tag{28}$$

of complexes of Nisnevich sheaves for all $n \in \mathbb{N}$.

Proof. — Denote by $K_0^{\oplus}$ the functor

$$\begin{aligned} K_0^{\oplus}(n) : \mathbf{Sm}/k \to & \mathbf{Ab} \\ X \mapsto & K_0^{\oplus}(X, \mathbb{G}_m^{\wedge n}). \end{aligned}$$

There is a natural morphism $f_n : K_0^{\oplus}(n) \to K_0^{\oplus,K,G}(n)$ given by

$$f_n(X) : [\mathcal{F}] \mapsto [\mathcal{F} \otimes_k K]$$

for any $X \in \mathbf{Sm}/k$, $[\mathcal{F}] \in K_0^{\oplus}(n)(X)$ and G acts on $\mathcal{F} \otimes_k K$ via its action on K. This yields a map of complex

$$C^*(f_n) : C^* K_0^{\oplus}(n) \to C^* K_0^{\oplus,K,G}(n),$$

hence a map $(C^* K_0^{\oplus}(n))_{Nis}[-n] \to \mathbb{Z}^{K,G}_{Gr}(n)$.

However, $(C^* K_0^{\oplus}(n))_{Nis}[-n]$ is canonically quasi-isomorphic to $\mathbb{Z}(n)$ as complexes of Nisnevich sheaves (cf. [**Sus03**, Theorem 6.1]). We obtain therefore a map

$$F_n : \mathbb{Z}(n) \to \mathbb{Z}^{K,G}_{Gr}(n).$$

We also have a natural map

$$\pi : \mathbb{Z}^{K,G}_{Gr}(0) \to \mathbb{Z}^{K,G}_{Gr}(n).$$

This yields a map

$$\pi \otimes F_n : \mathbb{Z}^{K,G}_{Gr}(0) \otimes \mathbb{Z}(n) \to \mathbb{Z}^{K,G}_{Gr}(n) \tag{29}$$

(see also section 3.2).

The natural transformation

$$K_0^{\oplus,K,G}(0) \to K_0^{K,G}$$

gives rise a morphism

$$\mathbb{Z}^{K,G}_{Gr}(0) \to \mathbb{Z}^{K,G},$$

hence a morphism

$$\alpha : \mathbb{Z}^{K,G}_{Gr}(0) \otimes \mathbb{Z}(n) \to \mathbb{Z}^{K,G} \otimes \mathbb{Z}(n) = \mathbb{Z}^{K,G}(n).$$

By Proposition 4.6 (see below), the map α is an isomorphism, so the composition

$$\phi_n := (\pi \otimes F_n) \circ \alpha^{-1} : \mathbb{Z}^{K,G}(n) \to \mathbb{Z}^{K,G}_{Gr}(n)$$

gives the desired morphism. □

The main goal of this chapter is to prove that ϕ_n are quasi-isomorphisms for every $n \in \mathbb{N}$. We refer n as the *weight* of these complexes.

4.1. Comparison of weight 0

If $X = \mathrm{Spec}R$ for a commutative k-algebra R, the (left) action of G on X induces a (right) action of G on the ring R, which we will write as

$$(r,g) \rightarrowtail r^g.$$

Let $R^{tw}[G] := \oplus_{g\in G} Rg$. The multiplication given by

$$(r_g.g)(r_h.h) := r_g r_h^{g^{-1}}.gh$$

for all $g, h \in G$ and $r_g, r_h \in R$ giving a ring structure on $R^{tw}[G]$. The ring $R^{tw}[G]$ is called the *twisted group ring* of G.

Lemma 4.2. — *Let $X = \mathrm{Spec}\ R$ be a noetherian affine G-scheme then the category of finitely generated left $R^{tw}[G]$-modules is equivalent to $\mathcal{M}_{G,X}$.*

Proof. — This is more or less a definition (see [**LS08**, Lemma 1.2]). □

Lemma 4.3. — *Let $X = \mathrm{Spec}\ R$ be a noetherian affine G-scheme with $\frac{1}{|G|} \in R$ then the category of finitely generated projective $R^{tw}[G]$-modules is equivalent to $\mathcal{P}_{G,X}$.*

Proof. — For any R-module M we set $M^{tw}[G] := R^{tw}[G] \otimes_R M$ where its elements may be written as $\Sigma_{g\in G} g.m_g$, $m_g \in M$. Let G acts diagonally on $R^{tw}[G] \otimes_R M$. It is easy to see that if M is a projective R-module, hence $M^{tw}[G]$ is a projective $R^{tw}[G]$-module.

If M is a G-module on R then by the previous lemma, M is a $R^{tw}[G]$-module and the projection $p : M^{tw}[G] \to M$ is a morphism of $R^{tw}[G]$-modules. If $|G|$ is invertible in R, let $i : M \to M^{tw}[G]$ is the $R^{tw}[G]$-morphism sending m to $\frac{1}{|G|}\Sigma_{g\in G} g \otimes m$. It is obvious that $p \circ i = \mathrm{id}_M$, therefore, M is a direct summand of $M^{tw}[G]$. Therefore if M is a projective R-module then it is a projective $R^{tw}[G]$-module. □

Proposition 4.4. — (Maschke) *Let $X = \mathrm{Spec}R$ be a noetherian affine G-scheme such that $\frac{1}{|G|} \in R$. Then, every short exact sequence of G-vector bundles on X splits.*

Proof. — Let

$$0 \to V' \to V \to V'' \to 0$$

be a short exact sequence of G-vector bundles on X. It corresponds to a short exact sequence of G-equivariant projective R-modules

$$0 \to P' \to P \to P'' \to 0 \tag{30}$$

Since P'' is projective R-module, the sequence (30) splits. Let $\pi' : P \to P'$ be the splitting map and consider P' as a G-R-submodule of P, we set

$$\pi := \frac{1}{|G|} \sum_{g\in G} g \circ \pi' \circ g^{-1} : P \to P.$$

If $r \in R$ and $p \in P$ then

$$\begin{aligned}
\pi(r.p) &= \frac{1}{|G|}\sum_{g\in G} g\circ\pi'\circ g^{-1}(r.p)\\
&= \frac{1}{|G|}\sum_{g\in G} g\circ\pi'[g^{-1}(r).g^{-1}(p)]\\
&= \frac{1}{|G|}\sum_{g\in G} g[g^{-1}(r).\pi'(g^{-1}(p))]\\
&= \frac{1}{|G|}\sum_{g\in G} r.g\circ\pi'\circ g^{-1}(p)\\
&= r.\pi(p).
\end{aligned}$$

This implies that π is an R-endomorphism of P.

For any $h \in G$ and $p \in P$, we have

$$\begin{aligned}
h\pi(p) &= h[\frac{1}{|G|}\sum_{g\in G} g\circ\pi'\circ g^{-1}(p)]\\
&= \frac{1}{|G|}\sum_{g\in G} hg\circ\pi'\circ g^{-1}(p)\\
&= \frac{1}{|G|}\sum_{g\in G} hg\circ\pi'\circ (hg)^{-1}(hp)\\
&= \frac{1}{|G|}\sum_{g\in G} g\circ\pi'\circ g^{-1}(hp)\\
&= \pi(hp).
\end{aligned}$$

This implies that π is a G-R-endomorphism of P.

Since $\pi'\circ g^{-1}p \in P'$, we have $g\circ\pi'\gamma^{-1}p \in P'$. Therefore π maps P into P'.

If $p \in P'$ then $\pi p = p$, so π is a projection. Denote by Q be the kernel of π. It is obvious that $p^0\circ g = g\circ p^0$ for all $g \in G$. If $p \in Q$ then $\pi(gp) = g(\pi p) = 0$, so $gp \in Q$. It implies that $Q \subset P$ is a G-invariant R-submodule of P and we obtain a short exact sequence of G-invariant projective R-modules

$$0 \to P' \leftrightarrows P \to P'' \to 0.$$

□

Corollary 4.5. — *Let G be a finite group acting on an affine G-scheme X. Assume that $\frac{1}{|G|} \in \mathcal{O}_X$ then $K_0^{\oplus}(G,X) \xrightarrow{\sim} K_0(G,X)$.*

Proposition 4.6. — *$\mathbb{Z}_{Gr}^{K,G}(0)$ is isomorphic to $\mathbb{Z}^{K,G}(0)$ in the derived category of Zariski sheaves on $\mathbf{Sm}/k$.*

Proof. — We only need to show that for any affine local smooth scheme X over k, the natural morphism of complexes

$$K_0(G, X\times_k K) \cong K_0^{\oplus}(G, X\times_k K) \to K_0^{\oplus}(G, X\times_k K\times\Delta^*)$$

is an quasi-ismorphism. This is the case because we have

$$K_0^{\oplus}(G, X \times_k K \times \Delta^*) \cong K_0(G, X \times_k K \times \Delta^*) \cong K_0(G, X \times_k K),$$

where first isomorphism comes from Corollary 4.5 and the last isomorphism is the consequence of homotopy invariance for equivariant K-theory over $\mathbf{Sm}/k$. $\square$

Suslin remarked that if F is a homotopy invariant K_0-presheaf then F has a canonical structure of a Zariski sheaf with transfers (cf. [**Sus03**, Remark 1.4.1]). But it seems to be long and not necessary for our purpose. We are going to prove directly that $\mathbb{Z}^{K,G}$ is a homotopy invariant presheaf with transfers.

4.2. Birationality

Recall that a presheaf of abelian groups F on $\mathbf{Sm}/k$ is called birational if for any X smooth and $U \subset X$ open dense subset, the restriction $F(X) \to F(U)$ is an isomorphism. In this section, we are going to prove the following

Theorem 4.7. — *$\mathbb{Z}^{K,G}$ is a sheaf with transfers, that is birational and homotopy invariant.*

The proof will be divided into several steps

Lemma 4.8. — *$\mathbb{Z}^{K,G}$ is a birational sheaf.*

Proof. — By localization property for equivariant G-theory we have for given $Y \in \mathbf{Sm}/k$, a closed subscheme $Z \subset Y$ and $U = Y \backslash Z$ a long exact sequence

$$\to G_0(G, Z \times_k K) \to G_0(G, X \times_k K) \to G_0(G, U \times_k K) \to 0.$$

Let $\mathcal{Z}$ be the family of all closed subschemes of Y of codimension ≥ 1, we have:

$$\varinjlim_{Z \in \mathcal{Z}} G_0(G, Z \times_k K) \xrightarrow{\phi} G_0(G, Y \times_k K) \to G_0(G, K(X) \times_k K) \to 0.$$

We will show that the map ϕ is 0 when Y is a spectrum of a local ring. The argument is a slight modification of Quillen's trick in [**Qui73**] where he shows that Gersten's conjecture for G-theory is true over regular semi-local rings containing a field.

We can assume that $Y = \mathrm{Spec}\mathcal{O}_{X,x}$ where X is an affine smooth scheme and $x \in X$ is a point. Let F be a G-coherent sheaf on $X \times_k K$ with support in $Z \times_k K$, $\mathrm{codim}_X Z \geq 1$ and D be a divisor on X containing Z. If we set $n := \dim X - 1$, then there is a morphism

$$\pi : X \to \mathbb{A}^n_k$$

such that the induced map $\bar{\pi} : D \to \mathbb{A}^n_k$ is finite and π is smooth on a neighborhood of x in X.

Consider the Cartesian diagram

$$\begin{tikzcd} D \times_{\mathbb{A}^n} X \arrow[r, "p"] \arrow[d, "q"] & X \arrow[d, "\pi"] \\ D \arrow[r, "\bar{\pi}"] & \mathbb{A}^n_k. \end{tikzcd} \tag{31}$$

The map q admits a section s, such that $p\circ s : D \to X$ is the natural inclusion. Clearly, p is finite.

Since π is smooth near x, $s(D)$ is a Cartier divisor on a neighborhood of $p^{-1}(x)$. Since p is finite, there is a neighborhood of x in X such that $s(D)$ is principal on $U' := p^{-1}(U)$. We may choose $U = X_f$ for some f not vanished at x.

Let t be a defining equation of $s(D)$ on U'. Taking the fibered product of (31) with K over k, we have the commutative diagram

$$\begin{tikzcd} D \times_{\mathbb{A}^n} X \times_k K \arrow[r, "p"] \arrow[d, "q"] & X \times_k K \arrow[d, "\pi"] \\ D \times_k K \arrow[r, "\bar{\pi}"] & \mathbb{A}^n_K. \end{tikzcd} \tag{32}$$

Let $D_U := D \cap U$, we have

$$\begin{tikzcd} D_U \times_{\mathbb{A}^n} X \times_k K \arrow[r, "p|U"] \arrow[d, "q"] & U \times_k K \\ D_U \times_k K \arrow[ur, "i"] & \end{tikzcd}$$

Let $\mathcal{G}$ be the restriction of F to $D_U \times_k K$, we have the exact sequence

$$0 \to p_*q^*\mathcal{G} \xrightarrow{p_*(\times t)} p_*q^*\mathcal{G} \to i_*\mathcal{G} \to 0.$$

on $U \times_k K$. Therefore $[i_*\mathcal{G}] = 0$ in $K_0(G, U \times_k K)$.

Since X is affine one has D is affine and F corresponds to G-module M over $X\times_k K$. If U has the form X_f then $i_*\mathcal{G}$ corresponds to M_f. It implies that $[M_f] = 0$ in $G_0(G, X_f \times_k K)$.

We have

$$G_0(G, \mathcal{O}_{X,x} \times_k K) = \lim_{f\in V} G_0(G, X_f \times_k K).$$

where V is the set of $f \in \mathcal{O}_X$ not vanishing at x. So $[M] = 0$ in $G_0(G, \mathcal{O}_{X,x} \times K)$. Therefore, the map ϕ is 0.

As a consequence, we have Gersten resolution for $\mathbb{Z}^{K,G}$. For any smooth scheme $X \in \mathbf{Sm}/k$ with generic points η_i, remark that K-theory and G-theory are the same for smooth schemes, we have an exact sequence

$$0 \to \mathbb{Z}^{K,G}(X) \to \bigoplus_i K_0^{K,G}(k(\eta_i)) \to 0$$

i.e., $\mathbb{Z}^{K,G}(X) \xrightarrow{\sim} \bigoplus_i K_0^{K,G}(k(\eta_i))$. Hence $\mathbb{Z}^{K,G}$ is birational sheaf. □

Remark 4.9. — *Since every birational presheaf is Nisnevich sheaf (see Proposition 1.14),* $\mathbb{Z}^{K,G}$ *is a Nisnevich sheaf.*

Lemma 4.10. — $\mathbb{Z}^{K,G}$ *is a sheaf with transfers.*

Proof. — Since $(|G|, \mathrm{char} k) = 1$, the twisted group ring $K^{tw}[G]$ is a semi-simple separable k-algebra (see [**LS08**]). By Artin- Wedderburn theorem

$$K^{tw}[G] = \prod_{finite} M_{n_i}(D_i)$$

where D_i are finite dimensional division algebras over k and $M_{n_i}(D_i)$ are the algebras of $n_i \times n_i$ matrices over D_i. Let K_i be the center of D_i then K_i/k is finite separable field extension.

For any separable field extension F/k (it is always the case when k is a perfect field), we have

$$\mathbb{Z}^{K,G}(F) = K_0^{K,G}(F) = K_0(G, F \otimes_k K) = K_0((F \otimes_k K)^{tw}[G]) = K_0(F \otimes_k K^{tw}[G]).$$

$$\begin{aligned} K_0(F \otimes_k K^{tw}[G]) =& K_0(F \otimes_k \prod M_{n_i}(D_i)) \\ =& K_0(\prod M_{n_i}(F \otimes_k D_i)) \\ =& \bigoplus K_0(M_{n_i}(F \otimes D_i)) \\ =& \bigoplus K_0(F \otimes D_i) \end{aligned}$$

where the last equality is a consequence of Morita equivalence. If $F \otimes_k K_i = \prod F_{ij}$ where F_{ij} are fields then they are finite over F and $F \otimes_k D_i = F \otimes_k K_i \otimes_{K_i} D_i = (\prod F_{ij}) \otimes_{K_i} D_i = \prod(F_{ij} \otimes_{K_i} D_i)$.

Since D_i is division algebra over k with center K_i, D_i is a central simple algebra over K_i and hence $F_{ij} \otimes_{K_i} D_i$ is a central simple algebra over F_{ij}. Therefore, we have

$$\begin{aligned} K_0(F \otimes_k K^{tw}[G]) = \bigoplus_{i,j} K_0(F_{ij} \otimes_{F_i} D_i) =& \bigoplus_{ij} K_0(M_{n_{ij}}(D_{ij})) \\ =& \bigoplus_{ij} K_0(D_{ij}) = \bigoplus_{i,j} \mathbb{Z} \end{aligned}$$

where n_{ij} are integer numbers and D_{ij} are division algebras with centers F_{ij}.

We set

$$\kappa_F = \rho_F \circ Nrd_F : K_0(F \otimes_k K^{tw}[G]) \xrightarrow{Nrd_F} \bigoplus K_0(F_{ij}) \xrightarrow{\rho_F} K_0(F)$$

where Nrd_F is defined to be the sum of maps

$$Nrd_{F_{ij}} : K_0(F_{ij} \otimes_{K_i} D_i) \to K_0(F_{ij})$$

that send generator of $K_0(F_{ij} \otimes_{K_i} D_i)$ to $e_{ij}[F_{ij}]$ where e_{ij} is the index of $F_{ij} \otimes_{K_i} D_i/F_{ij}$ (i.e., $e_{ij}^2 = \deg[D_{ij} : F_{ij}]$) and

$$\rho_F : \bigoplus_{ij} K_0(F_{ij}) = \bigoplus K_0(F \otimes_k K_i) \to K_0(F).$$

is the sum of push-forward maps.

If L/F is a separable field extension and $f : \mathrm{Spec}L \to \mathrm{Spec}F$ be the structure map, we have

$$\begin{aligned}
K_0(G, L \otimes_k K) =& K_0(L \otimes_k K^{tw}[G]) \qquad \text{(by Lemma 4.3)}\\
=& K_0(L \otimes_F F \otimes K^{tw}[G])\\
=& \bigoplus K_0(L \otimes_F D_{ij})\\
=& \bigoplus K_0(L \otimes_F F_{ij} \otimes_{F_{ij}} D_{ij})\\
=& \bigoplus_{i,j,k} K_0(L_{ijk} \otimes_{F_{ij}} D_{ij}) \text{ where } \prod_k L_{ijk} = L \otimes_F F_{ij}\\
=& \bigoplus_{i,j,k} K_0(M_{n_{ijk}}(D_{ijk}))
\end{aligned}$$

where D_{ijk} is a division algebra with center L_{ijk}

(1) If L/F is finite separable then the push-forward

$$f_* : K_0^{K,G}(L) \to K_0^{K,G}(F) \tag{33}$$

sends $[D_{ijk}]$ to $\deg[L_{ijk} : L_{ij}][D_{ij}]$.

(2) If L/F is any separable field extension then the pull-back

$$f^* : K_0^{K,G}(F) \to K_0^{K,G}(L) \tag{34}$$

sends $[D_{ij}]$ to $\sum_k n_{ijk}[D_{ijk}]$.

If $f : Y' \to Y$ be a morphism in $\mathbf{Sm}/k$, we are going to show that the pull-back

$$f_1^* : \mathbb{Z}^{K,G}(Y) \to \mathbb{Z}^{K,G}(Y')$$

is the same with the pull-back

$$f_2^* : K_0(G, K(Y) \otimes_k K) \to \bigoplus_{\eta \in Y'^{(0)}} K_0(G, k(\eta) \otimes_k K)$$

defined in (34), i.e., they are induced from the natural homomorphism

$$f^* : K_0(G, Y \times_k K) \to K_0(G, Y' \times_k K).$$

Let $\mathcal{M}^G_{(1)}(Y \times_k K)$ be the category of coherent G-sheaves on $Y \times_k K$ whose support containing no generic point. Set

$$G_0^{(1)}(G, Y \times_k K) := K_0(\mathcal{M}^G_{(1)}(Y \times_k K)),$$

then by localization property ([**Qui73**]), we have an exact sequence

$$G_0^{(1)}(G, Y \times_k K) \to G_0(G, Y \times_k K) \to G_0(G, K(Y) \otimes_k K) \to 0.$$

We also have a well-defined map

$$f^* : G_0(G, Y \times_k K) \to G_0(G, Y' \times_k K).$$

Using Quillen's trick again (as in the proof of Lemma 4.10), we see that the composition

$$G_0^{(1)}(G, Y \times_k K) \to G_0(G, Y \times_k K) \xrightarrow{f^*} G_0(G, Y' \times_k K) \to G_0(G, K(Y') \otimes_k K)$$

is 0. This yields a well-defined map

$$\phi : G_0(G, K(Y) \otimes_k K) \to G_0(G, K(Y') \otimes_k K).$$

We have already identified $\mathbb{Z}^{K,G}(Y)$ with $G_0(G, K(Y)\otimes_k K)$ and $\mathbb{Z}^{K,G}(Y')$ with $G_0(G, K(Y')\otimes_k K)$ in the same way. Therefore $f_1^* = f_2^* = \phi$.

Similarly, if $f : Z \to X$ is a finite and surjective over X then the push-forward

$$f_{1*} : \mathbb{Z}^{K,G}(Z) \to \mathbb{Z}^{K,G}(X)$$

is the same with

$$f_{2*} : K_0(G, K(Z) \otimes_k K) \to K_0(G, K(X) \otimes_k K)$$

defined in (33), i.e., they are induced from the natural morphism

$$f_* : G_0(G, Z \times_k K) \to G_0(G, X \times_k K).$$

Indeed, we have f_* maps $G_0^{(1)}(G, Z \times_k K)$ to $G_0^{(1)}(G, X \times_k K)$ because f is finite, hence the composition

$$G_0^{(1)}(G, Z \times_k K) \to G_0(G, Z \times_k K) \xrightarrow{f_*} G_0(G, X \times_k K) \to G_0(G, K(X) \otimes_k K)$$

is 0. This yields a map

$$\theta : G_0(G, K(Z) \otimes_k K) \to G_0(G, K(X) \otimes_k K)$$

and we have $f_{1*} = \theta = f_{2*}$.

<u>Claim 1</u>: The assignment $L \to Nrd_L$ defines a morphism of sheaves on $\mathbf{Sm}/k$

$$Nrd : \mathbb{Z}^{K,G} \to \bigoplus_i \mathbb{Z}_{K_i}$$

and $\mathbb{Z}^{K,G}$ becomes a subsheaf of $\bigoplus_i \mathbb{Z}_{K_i}$, where $\mathbb{Z}_{K_i} := \mathbb{Z}_{tr}(\mathrm{Spec}K_i)$, the presheaf with transfers represented by $\mathrm{Spec}K_i$.

<u>Proof of the claim 1</u>

We only need to show that if L/F is a separable field extension, the diagram

$$\begin{array}{ccc} \mathbb{Z}^{K,G}(F) & \xrightarrow{Nrd_F} & \bigoplus_i \mathbb{Z}_{K_i}(F) \\ \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle f^*} \\ \mathbb{Z}^{K,G}(L) & \xrightarrow{Nrd_L} & \bigoplus_i \mathbb{Z}_{K_i}(L). \end{array}$$

commutes.

Indeed, if $F \otimes_k K_i = \prod F_{ij}$ as a product of fields then Nrd_F is defined by the direct sum of maps

$$Nrd_{F_{ij}} : K_0(F_{ij} \otimes_{K_i} D_i) \to K_0(F_{ij}),$$

sends the generator of $K_0(F_{ij} \otimes_{K_i} D_i)$ to $e_{F_{ij}}[F_{ij}]$, where $e_{F_{ij}}$ is the index of $F_{ij} \otimes_{K_i} D_i/F_{ij}$. Similarly, if $L \otimes_k K_i = L \otimes_F F \otimes_k K_i = L \otimes_F \prod F_{ij} = \prod L_{ijk}$ then Nrd_L is defined by the direct sum of maps

$$Nrd_{L_{ij}} : K_0(L_{ijk} \otimes_{K_i} D_i) \to K_0(L_{ijk}),$$

which sends the generator of $K_0(L_{ijk} \otimes_{K_i} D_i)$ to $e_{L_{ijk}}[L_{ijk}]$ where $e_{L_{ijk}}$ is the index of $L_{ijk} \otimes_{K_i} D_i/L_{ijk}$. So, we only need to prove that for given i, j, k, the following diagram

$$\begin{array}{ccc} K_0(F_{ij} \otimes_{K_i} D_i) & \xrightarrow{Nrd_{F_{ij}}} & K_0(F_{ij}) \\ \Big\downarrow f^* & & \Big\downarrow f^* \\ K_0(L_{ijk} \otimes_{K_i} D_i) & \xrightarrow{Nrd_{L_{ijk}}} & K_0(L_{ijk}) \end{array}$$

commutes. In this case D_i is a central simple algebra over K_i and this diagram commutes by Lemma 5.2.1 [**KL10**]. We easily see that $\mathbb{Z}^{K,G}$ is a subsheaf of $\bigoplus_i \mathbb{Z}_{K_i}$. This completes the claim 1.

If $X, Y \in \mathbf{Sm}/k$. Let $Z \subset X \times_k Y$ be an integral subscheme which is finite over X and surjective onto a component of X.

Let $p : Z \to X,\ q : Z \to Y$ be the maps induced by projections. Define

$$Z^* : \mathbb{Z}^{K,G}(Y) \to \mathbb{Z}^{K,G}(X)$$

by $Z^* : p_* \circ q^*$, extend this operation to $Cor_k(X, Y)$ by linearity.

<u>Claim 2</u>: For $Z_1 \in Cor_k(X, Y),\ Z_2 \in Cor_k(Y, Z)$, we have $(Z_2 \circ Z_1)^* = Z_1^* \circ Z_2^*$.

<u>Proof of Claim 2</u>:

If L/F is a finite separable field extension, by computing the degree, we have

$$\begin{array}{ccccc} \mathbb{Z}_0^{K,G}(L) & \xrightarrow{Nrd_L} & \bigoplus \mathbb{Z}_{K_i}(F) & = & \bigoplus K_0(L \otimes_k K_i) \\ \Big\downarrow f_* & & & & \Big\downarrow f_* \\ \mathbb{Z}^{K,G}(F) & \xrightarrow{Nrd_F} & \bigoplus \mathbb{Z}_{K_i}(F) & = & \bigoplus K_0(F \otimes_k K_i) \end{array}$$

commutes. Hence, f_* commutes with Nrd.

From the proof of Claim 1, we see that Nrd commutes with f^* for any separable field extension. Therefore, the action of correspondence Z for $\mathbb{Z}^{K,G}$ and for $\mathbb{Z}_{K_i}$ commutes with $Nrd^{K,G}$. Since Nrd is injective on $\mathbf{Sm}/k$, this imply that $(Z_2 \circ Z_1)^* = Z_1^* \circ Z_2^*$.

□

Remark 4.11. — *The assumption $(|G|, \mathrm{char} k) = 1$ is crucial in the proof. Indeed, if $|G|$ is divided by the characteristic of k and G acts trivially on k then $h := \Sigma_{g \in G} g$ is a non-zero element in $k[G]$ satisfying $h^2 = 0$. This implies that $k[G]$ is not semi-simple.*

By the argument in this proof, we obtain natural maps of sheaves with transfers

$$\mathbb{Z}^{K,G} \xrightarrow{Nrd} \bigoplus_i \mathbb{Z}_{K_i} \to \mathbb{Z}.$$

Nrd is injective but the composition is not. This composition is an isomorphism when $G = Gal(K/k)$. It comes from the fact that when $G = Gal(K/k)$, the natural morphism $K(X) \to K(G, X \times_k K)$ is homotopy equivalent.

By Proposition 1.14, we have

Corollary 4.12. — *$\mathbb{Z}^{K,G}$ is a birational motivic sheaf, hence $\mathbb{Z}^{K,G}(n)[2n]$ is well-connected for every $n \in \mathbb{N}$.*

4.3. Rationally contractible presheaves

Let $X \in \mathbf{Sm}/k$ be a smooth scheme and $\{X_i\}_{i=0}^n$ be a family of closed subschemes in X such that for any subset $I \subset \{0, \dots, n\}$, the intersection $\cap_{i \in I} X_i$ is smooth. The *relative complex* $\mathbb{Z}_{Zar}(X; \{X_i\}_{i=0}^n)$ is the complex of Zariski sheaves

$$\dots \to \oplus_{i_1 < \dots < i_k} \mathbb{Z}_{Zar}(X_{i_1} \cap \dots \cap X_{i_k}) \to \dots \to \oplus_i \mathbb{Z}_{Zar}(X_i) \to \mathbb{Z}_{Zar}(X)$$

where $\mathbb{Z}_{Zar}(X_{i_1} \cap \dots \cap X_{i_k})$ stands in degree $-k$ and the differential is the alternative sum of maps induced by inclusion.

If F^* is any complex of Zariski sheaves, the *polyrelative cohomology* of X with respect to $\{X_i\}_{i=0}^n$ is define by

$$H^p_{Zar}(X; \{X_i\}_{i=0}^n, F^*) := \mathrm{Hom}_{D(\mathrm{Shv}_{Zar}(k))}(\mathbb{Z}_{Zar}(X; \{X_i\}_{i=0}^n), F^*[p]).$$

Replace Zariski topology by Nisnevich counterpart, we define

$$H^p_{Nis}(X; \{X_i\}_{i=0}^n, F^*) := \mathrm{Hom}_{D(\mathrm{Shv}_{Nis}(k))}(\mathbb{Z}_{Nis}(X; \{X_i\}_{i=0}^n), F^*[p]).$$

The main relative complexes we are interested in are

$$\mathbb{Z}_{tr}(\Delta^n_E, \{\partial_i \Delta^n_E\}_{i=0}^n)$$

and

$$\mathbb{Z}_{tr}(\hat{\Delta}^n_E, \{\partial_i \Delta^n_E\}_{i=0}^n)$$

where E/k is a finitely field extension, $\partial_i \Delta^n_E \subset \Delta^n_E$ is the n-th face and $\partial_i \hat{\Delta}^n_E := \partial_i \Delta^n_E \cap \hat{\Delta}^n_E$. The subscript *tr* refers the complex as a complex of sheaves with transfers, without mentioning about topology.

Remark 4.13. — *The polyrelative cohomology is a generalization of relative cycle complex studied by Geisser-Levine in* [**GL00**], *where they showed that the relative cycle complexes of* Δ^n_E ($\hat{\Delta}^n_E$) *with respect to* $\{\partial_i \Delta^n_E\}_{i=0}^n$ ($\{\partial_i \Delta^n_E \cap \hat{\Delta}^n_E\}_{i=0}^n$, *respectively*) *are useful tools to understand motivic cohomology (more precisely, higher Chow groups of Bloch). These methods turn out to be very useful to study Grayson cohomology as well.*

Let $F : \mathbf{Sm}/k \to \mathbf{Ab}$, denote by $\widehat{C_1} F$ the presheaf defined by

$$\widehat{C_1} F(X) = \varinjlim_U F(U),$$

where U runs over all open subschemes of $X \times \mathbb{A}^1$ containing $X \times \{0, 1\}$. The restrictions i_i to $X \times \{i\}$ induce $i_i^* : \widehat{C_1} F \to F$.

Definition 4.14. — *A presheaf F is called rationally contractible if there exist a natural transformation $s : F \to \widehat{C_1} F$ such that $i_0^* \circ s = 0$ and $i_1^* \circ s = id_F$.*

Lemma 4.15. — *If F is a rationally contractible presheaf then the complex $F(\widehat{\Delta}^*)$ is contractible.*

Proof. — Is is easy to see that $C^n F$ is also rationally contractible for all $n \in \mathbb{Z}$. We use the standard simplicial decomposition of the polyhedron $\Delta^1 \times \Delta^n$ as in the proof of Lemma 3.1 and the transformation s to construct a chain homotopy from $\mathrm{id} : F(\widehat{\Delta}^*) \to F(\widehat{\Delta}^*)$ to the zero map. $\square$

Lemma 4.16. — ([**Sus03**, Theorem 2.7]) *If F is a rationally contractible $K_0^{\oplus}$-presheaf then the complex $C^*(F)(\widehat{\Delta}^n, \{\partial_i\widehat{\Delta}^n\})$ is acyclic in positive degrees and hence*

$$H^p_{Zar}(\widehat{\Delta}^n, \{\partial_i\widehat{\Delta}^n\}, C^*(F)_{Zar}) = H^p_{Nis}(\widehat{\Delta}^n, \{\partial_i\widehat{\Delta}^n\}, C^*(F)_{Nis}) = 0$$

for all $p \geq 0$.

Proposition 4.17. — *The presheaf $K_0^{\oplus}(G, - \times_k K, \mathbb{G}_m^{\wedge n})$ is rationally contractible. As a consequence, $\forall n, m \geq 0$, $p \geq n$ and any field extension E/k*

$$H^p(\widehat{\Delta}^m_E, \{\partial_i\widehat{\Delta}^m_E\}^m_{i=0}, \mathbb{Z}^{K,G}_{Gr}(n)) = 0.$$

Proof. — We consider the case when $n = 1$, other cases are proved similarly.

Let $Z \subset \mathbb{G}_m \times \mathbb{A}^1$ be the closed subscheme defined by

$$W := \{(x,t)|t.x + (1-t).e = 0\}.$$

We set $U := \mathbb{G}_m \times \mathbb{A}^1 \backslash Z$ together with the map

$$f : U \to \mathbb{G}_m,\ (x,t) \mapsto t.x + (1-t).e.$$

For any $F \in \mathcal{P}(G, Y, \mathbb{G}^1)$ let $W := \mathrm{Supp}F \subset Y \times \mathbb{G}_m$ then W is finite and flat over Y. Therefore $F \boxtimes 1_{\mathbb{A}^1}$ is finite and flat over $Y \times \mathbb{A}^1$ whose support is $W \times \Delta_{\mathbb{A}^1}$ where $\Delta_{\mathbb{A}^1}$ is the diagonal of $\mathbb{A}^1 \times \mathbb{A}^1$.

Let $S := W \times \Delta_{\mathbb{A}^1} \cap Z \times Y \times \mathbb{A}^1$ be the closed subset of $Y \times \mathbb{A}^1 \times \mathbb{G}^1 \times \mathbb{A}^1$ then it is finite and flat over $Y \times \mathbb{A}^1$. It is obvious that S does not contain any point whose $\mathbb{A}^1$ factors are 0 or 1. We set $T \subset Y \times \mathbb{A}^1$ is the push-forward

$$T := p_{Y \times \mathbb{A}^1,*}(S)$$

and $V := Y \times \mathbb{A}^1 \backslash T$. Clearly $Y \times \{0,1\} \subset V$.

Let $H := i^*(F \boxtimes \Delta_{\mathbb{A}^1})$ be the pull-back of $F \boxtimes \Delta_{\mathbb{A}^1}$ along the open embedding $V \times \mathbb{G}_m\mathbb{A}^1 \xrightarrow{i} Y \times \mathbb{A}^1\mathbb{G}_m\mathbb{A}^1$ then $\mathrm{Supp}H \subset V \times U$. Pushing-forward H along the map $f : U \to \mathbb{G}_m$ to obtain an element $F' \in \mathcal{P}(G, V, \mathbb{G}_m)$. It is clear that $F \mapsto F'$ preserves split short exact sequence, hence we obtain a morphism

$$\phi : K_0^{\oplus}(G, Y, \mathbb{G}_m) \to \varinjlim_{V} K_0^{\oplus}(G, V, \mathbb{G}_m)$$

with remark that if $F \in K_0^{\oplus}(G, Y, e)$ then $\phi(F) \in \varinjlim_V K_0^{\oplus}(G, V, \mathbb{G}_m)$. □

We have therefore a natural transformation

$$s : K_0^{\oplus}(G, -, \mathbb{G}_m^{\wedge 1}) \to \widehat{C_1}K_0^{\oplus}(G, -, \mathbb{G}_m^{\wedge 1})$$

By direct computation we have

$$i_0^* \circ s = 0, \text{ and } i_1^* \circ s = id_{K_0^{\oplus}(G, -, \mathbb{G}_m^{\wedge 1})},$$

hence $K_0^{\oplus}(G, -, \mathbb{G}_m^{\wedge 1})$ is rationally contractible presheaf.

Proposition 4.18. —

$$H^p(\widehat{\Delta}^m_E, \{\partial_i\widehat{\Delta}^m_E\}^m_{i=0}, \mathbb{Z}^{K,G}(n)) = 0$$

$\forall n, m \geq 0$, $p \geq n$ and any field extension E/k.

Proof. — This come from the fact that $\mathbb{Z}^{K,G}$ is birational motivic sheaf, hence $\mathbb{Z}^{K,G}(n)[2n]$ is well-connected. $\square$

4.4. Purity

Theorem 4.19. — *Grayson complexes $\mathbb{Z}^{K,G}_{Gr}(n)$ satisfy cancellation property, i.e., for any $X \in \mathbf{Sm}/k$ there is a canonical isomorphism*

$$H^{s-1}_{Nis}(X, \mathbb{Z}^{K,G}_{Gr}(n-1)) \simeq H^s_{Nis}(X \wedge \mathbb{G}_m, \mathbb{Z}^{K,G}_{Gr}(n))$$

defined as external multiplication by the element $\lambda = 1_{\mathbb{G}_m} - e \in H^1(\mathbb{G}_m^{\wedge 1}, \mathbb{Z}^{K,G}_{Gr}(1))$.

Proof. — Let $\pi : X \times \mathbb{G}_m \to X$ be the projection. By Leray spectral sequence

$$\begin{aligned} E_2^{p,q} &:= H^p(X, R^q\pi_*(C^*(K_0^{\oplus,K,G}(n)))_{Zar}|_{X\times\mathbb{G}_m}) \Rightarrow \\ &\Rightarrow H^{p+q}(X \times \mathbb{G}_m, (C^*(K_0^{\oplus,K,G}(n)))_{Zar}). \end{aligned} \tag{35}$$

where we use the notation $|_X$ to denote the restriction of a sheaf or a complex of sheaves onto the small Zariski site of X.

Denote by $\mathcal{H}^q$ the Zariski sheaf associated to the q-th cohomology presheaf $H^q(C^*(K_0^{\oplus,K,G}(n)))$, we have the hypercohomology spectral sequence

$$E_2^{p,q} := R^p\pi_*(\mathcal{H}^q|_{X\times\mathbb{G}_m}) \Rightarrow H^{p+q}(R\pi_*(C^*(K_0^{\oplus,K,G}(n)))_{Zar}|_{X\times\mathbb{G}_m}). \tag{36}$$

Since $H^q(C^*(K_0^{\oplus,K,G}(n)))$ is homotopy invariant $K_0^{\oplus}$-presheaf, the sheaf $\mathcal{H}^q$ is homotopy invariant pretheories by Lemma 1.21, hence on stalks

$$R^p\pi_*(\mathcal{H}^q|_{X\times\mathbb{G}_m})_x = H^p(X_x \times \mathbb{G}_m, \mathcal{H}^q) = 0$$

for $p > 0$. Thus $E_2^{p,q} = 0$ for $p \neq 0$, hence the spectral sequence (36) degenerates. We have

$$\pi_*(\mathcal{H}^q|_{X\times\mathbb{G}_m}) \cong H^q(R\pi_*(C^*(K_0^{\oplus,K,G}(n)))_{Zar}|_{X\times\mathbb{G}_m})$$

together with the identity

$$\mathcal{H}^q(X_x \times \mathbb{G}_m) = \mathcal{H}^q(X_x) \times \mathcal{H}^q_{-1}(X_x)$$

where $\mathcal{H}^q_{-1}(X) := \mathrm{Ker}(\mathcal{H}^q(X \times \mathbb{G}_m) \to \mathcal{H}^q(X))$ is the Voevodsky's contraction (cf. [**Voe00b**]).

If F is a homotopy invariant $K^{\oplus}$-presheaf then $(F_{-1})_{Zar} = (F_{Zar})_{-1}$ ([**Sus03**, Theorem 4.8]) and hence

$$\mathcal{H}^q_{-1}(X_x) = (H^q(C^*(K_0^{\oplus,K,G}(n)))_{-1})_{Zar}(X_x).$$

By Theorem

$$(H^q(C^*(K_0^{\oplus,K,G}(n)))_{-1})_{Zar}(X_x) = (H^q(C^*(K_0^{\oplus,K,G}(n-1))))_{Zar}(X_x).$$

Therefore, we have

$$\begin{aligned} &H^q(R\pi_*(C^*(K_0^{\oplus,K,G}(n)))_{Zar}|_{X\times\mathbb{G}_m}) = \\ &= H^q(C^*(K_0^{\oplus,K,G}(n))_{Zar}|_X) \oplus H^q(C^*(K_0^{\oplus,K,G}(n-1))_{Zar}|_X) \end{aligned}$$

i.e., the canonical homomorphism of complexes

$$(R\pi_*(C^*(K_0^{\oplus,K,G}(n)))_{Zar}|_{X\times\mathbb{G}_m}) = C^*(K_0^{\oplus,K,G}(n))_{Zar}|_X \oplus \\ \oplus\, C^*(K_0^{\oplus,K,G}(n-1))_{Zar}|_X$$

is a quasi-isomorphism and hence

$$H^q_{Zar}(X\times\mathbb{G}_m, C^*(K_0^{\oplus,K,G}(n))_{Zar}) = H^q_{Zar}(X, C^*(K_0^{\oplus,K,G}(n))_{Zar})\oplus \\ \oplus\, H^q_{Zar}(X, C^*(K_0^{\oplus,K,G}(n-1))_{Zar}).$$

This implies that

$$H^q_{Zar}(X\wedge\mathbb{G}_m, C^*(K_0^{\oplus,K,G}(n))_{Zar}) = H^q_{Zar}(X, C^*(K_0^{\oplus,K,G}(n-1))_{Zar})$$

Since both sides do not change if we replace Zariski topology by its Nisnevich counterpart, we obtain the desired result. □

Theorem 4.20. — (Cancellation theorem for "Motivic cohomology"). *There is a canonical isomorphism $H^{s-1}_{Nis}(X,\mathbb{Z}^{K,G}(n-1)) \simeq H^s_{Nis}(X\wedge\mathbb{G}_m,\mathbb{Z}^{K,G}(n))$ defined as external multiplication by the element $\lambda = 1_{\mathbb{G}_m} - e \in H^1(\mathbb{G}_m^{\wedge 1},\mathbb{Z}(1))$.*

We have the sequence of complexes $\mathbb{Z}^{K,G}_{Gr}(n)$ with the following properties

(1) The complexes $\mathbb{Z}^{K,G}_{Gr}(0)$ is canonically quasi-isomorphic to the locally constant sheaf $\mathbb{Z}^{K,G}$, positioned in degree 0.
(2) All the cohomology sheaves $\mathcal{H}^q(\mathbb{Z}^{K,G}_{Gr}(n))$ are strictly homotopy invariant.
(3) For all $i,j\geq 0$, we have pairings $\mathbb{Z}^{K,G}_{Gr}(i)\otimes^L\mathbb{Z}^{K,G}_{Gr}(j)\to\mathbb{Z}^{K,G}_{Gr}(i+j)$ that are associate and commutative. Moreover, we have the following commutative diagram in the derived category

$$\begin{array}{ccc} \mathbb{Z}\otimes^L\mathbb{Z}^{K,G}_{Gr}(n) & \xrightarrow{=} & \mathbb{Z}^{K,G}_{Gr}(n) \\ \downarrow & & \downarrow{=} \\ \mathbb{Z}^{K,G}\otimes^L\mathbb{Z}^{K,G}_{Gr}(n) & \longrightarrow & \mathbb{Z}^{K,G}_{Gr}(n). \end{array}$$

(4) There is a canonical cohomology class $\lambda\in H^1(\mathbb{G}_m^{\wedge 1},\mathbb{Z}^{K,G}_{Gr}(1))$ such that for any smooth scheme X, external multiplication by λ defines isomorphisms

$$H^{p-1}(X,\mathbb{Z}^{K,G}_{Gr}(n-1))\to H^p(X\wedge\mathbb{G}^m,\mathbb{Z}^{K,G}_{Gr}(n)).$$

Using the deformation to the normal cones, we have

Proposition 4.21. — *Let $Z\subset X$ be smooth subscheme of a smooth scheme X everywhere of codimension m. Then we have the following commutative diagram.*

$$\begin{array}{ccc} H^{s-2m}_{Nis}(Z,\mathbb{Z}^{K,G}_{Gr}(n-m)) & \xrightarrow{\sim} & H^s_Z(X,\mathbb{Z}^{K,G}(n)) \\ {\scriptstyle (f_{n-m})_*}\downarrow & & \downarrow{\scriptstyle (f_n)_*} \\ H^{s-2m}_{Nis}(Z,\mathbb{Z}^{K,G}_{Gr}(n-m)) & \xrightarrow{\sim} & H^s_Z(X,\mathbb{Z}^{K,G}_{Gr}(n)). \end{array}$$

Proof. — See [**Sus03**, Section 5]. □

4.5. Comparision for arbitrary weight

We are now in the situation to prove

Theorem 4.22. — *For all $n \geq 0$, the canonical homomorphism of complexes of Nisnevich sheaves*

$$f_n : \mathbb{Z}^{K,G}(n) \to \mathbb{Z}^{K,G}_{Gr}(n)$$

is a quasi-isomorphism.

Proof. — We will first assume that k is a perfect field and prove by induction on weight n.

The case $n = 0$ was proved in Section 4.1. Assume the statement is true for $0 \leq m < n$, we will prove that it is also true for $m = n$.

Since the complexes $\mathbb{Z}^{K,G}_{Gr}(n)$ and $\mathbb{Z}^{K,G}(n)$ have cohomology sheaves which are homotopy invariant pretheories, it suffices to show that for any finitely generated field extension F/k, the map

$$f_n^* : H^q_{Nis}(F, \mathbb{Z}^{K,G}(n)) \to H^q_{Nis}(F, \mathbb{Z}^{K,G}_{Gr}(n))$$

is an isomorphism for all n (Proposition 1.22).

We have the isomorphism

$$H^p(F, \mathbb{Z}^{K,G}(n)) = H^{m+p}(\Delta^m_F, \{\partial_i \Delta^m_F\}^m_{i=0}, \mathbb{Z}^{K,G}(n))$$

and similarly for $\mathbb{Z}^{K,G}_{Gr}(n)$.

Let $\mathcal{Z}$ denote the family of supports on Δ^m_F, consisting of all closed subschemes $Z \subset \Delta^m_F$ containing no vertices. Then we have the long exact sequence

$$\begin{aligned} \to H^{m+p-1}(\hat{\Delta}^m_F, \{\partial_i \hat{\Delta}^m_F\}^m_{i=0}, \mathbb{Z}^{K,G}_{Gr}(n)) \to H^{m+p}_{\mathcal{Z}}(\Delta^m_F, \{\partial_i \Delta^m_F\}^m_{i=0}, \mathbb{Z}^{K,G}_{Gr}(n)) \to \\ \to H^{m+p}(\Delta^m_F, \{\partial_i \Delta^m_F\}^m_{i=0}, \mathbb{Z}^{K,G}_{Gr}(n)) \to H^{m+p}(\hat{\Delta}^m_F, \{\partial_i \hat{\Delta}^m_F\}^m_{i=0}, \mathbb{Z}^{K,G}_{Gr}(n)) \to \end{aligned}$$

and the natural homomorphism from this exact sequence to a similar exact sequence for polyrelative cohomology with supports with coefficients in $\mathbb{Z}^{K,G}(n)$.

By Lemma 4.17, $H^{m+p}(\hat{\Delta}^m_F, \{\partial_i \hat{\Delta}^m_F\}^m_{i=0}, \mathbb{Z}^{K,G}_{Gr}(n)) = 0$ for $m + p \geq n$.

Taking m large enought, we have isomorphism

$$H^{m+p}_{\mathcal{Z}}(\Delta^m_F, \{\partial_i \Delta^m_F\}^m_{i=0}, \mathbb{Z}^{K,G}_{Gr}(n)) \xrightarrow{\sim} H^{m+p}(\Delta^m_F, \{\partial_i \Delta^m_F\}^m_{i=0}, \mathbb{Z}^{K,G}_{Gr}(n)).$$

and similar isomorphism for $\mathbb{Z}^{K,G}(n)$-cohomology.

The remaining task is to prove

$$H^{m+p}_{\mathcal{Z}}(\Delta^m_F, \{\partial_i \Delta^m_F\}^m_{i=0}, \mathbb{Z}^{K,G}(n)) \to H^{m+p}_{\mathcal{Z}}(\Delta^m_F, \{\partial_i \Delta^m_F\}^m_{i=0}, \mathbb{Z}^{K,G}_{Gr}(n))$$

is an isomorphism. Equivalently, we prove that for all closed subscheme $Z \subset \Delta^m_F$ of positive codimension, the homomorphism

$$H^{m+p}_{Z}(\Delta^m_F, \{\partial_i \Delta^m_F\}^m_{i=0}, \mathbb{Z}^{K,G}(n)) \to H^{m+p}_{Z}(\Delta^m_F, \{\partial_i \Delta^m_F\}^m_{i=0}, \mathbb{Z}^{K,G}_{Gr}(n))$$

is an isomorphism.

Indeed, if $Z \subset Z' \subset X$ are closed subschemes where $X \in \mathbf{Sm}/k$ then for any complex of Nisnevich sheaves F^* we have the long exact sequence

$$\ldots \to H^p_Z(X, F^*) \to H^p_{Z'}(X, F^*) \to H^p_{Z' \backslash Z}(X \backslash Z, F^*) \to H^{p+1}_Z(X, F^*) \to \ldots . \qquad (37)$$

This property also holds for polyrelative cohomology with supports by hypercohomology spectral sequence.

Since k is a perfect field, every $Z \subset \Delta^m$ admits a stratification with smooth strata, i.e., a sequence

$$\emptyset \subset Z_0 \subset Z_1 \subset \ldots \subset Z_k = Z$$

of closed subschemes of Z such that $Z_{i+1} \backslash Z_i$ are smooth for all $0 \leq i \leq k-1$. By (37), we can assume that Z is smooth.

By Proposition 4.21, we have the commutative diagram

$$\begin{array}{ccc} H^{s-2m}_{Nis}(Z, \mathbb{Z}^{K,G}_{Gr}(n-m)) & \xrightarrow{\sim} & H^s_Z(X, \mathbb{Z}^{K,G}(n)) \\ {\scriptstyle (f_{n-m})_*}\downarrow & & \downarrow{\scriptstyle (f_n)_*} \\ H^{s-2m}_{Nis}(Z, \mathbb{Z}^{K,G}_{Gr}(n-m)) & \xrightarrow{\sim} & H^s_Z(X, \mathbb{Z}^{K,G}_{Gr}(n)). \end{array}$$

where the morphism $(f_{n-m})_*$ is isomorphism by induction hypothesis. We conclude that $(f_n)_*$ is an isomorphism which prove the theorem when k is a perfect field.

We consider now the case k is not a perfect field. For any finitely field extension F/k the groups $H^*(F, \mathbb{Z}^{K,G}(n))$ and $H^*(F, \mathbb{Z}^{K,G}_{Gr}(n))$ are defined intrinsically in terms of the field F and are independent of the choice of the base field k. Assume that char$k = p > 0$ then

$$f_n^* : H^q_{Nis}(F, \mathbb{Z}^{K,G}(n)) \to H^q_{Nis}(F, \mathbb{Z}^{K,G}_{Gr}(n))$$

is an isomorphism for all F/k finitely generated field extension if and only if it is an isomorphism for all finitely generated field over $\mathbb{Z}/p$ (cf. [**Sus03**, Lemma 6.1.1]). However, any field of characteristic p may be written as a direct limit of fields finitely generated over $\mathbb{Z}/p$ and the above cohomology groups commute with direct limits. □

CHAPTER 5

COMPARISION OF SPECTRAL SEQUENCES

5.1. Comparision in $\mathcal{SH}_{S^1}(k)$

We recall the notations in Chapter 4 that G is a finite group acting on the field K with fixed subfield $k = K^G$ and $(|G|, \mathrm{char}(k)) = 1$. In this section, we will assume that k is an infinite perfect field.

Denote by E the presheaf of spectra given by

$$E := K^{K,G} : \mathbf{Sm}/k \to \mathbf{Spt}$$
$$Y \mapsto K(G, Y \times_k K)$$

where G acts on $Y \times_k K$ via its action on K.

Proposition 5.1. — [**LS08**, Proposition 5.1] *E is a well-connected theory as defined in Definition 1.9.*

We have therefore an identification

$$E^{(p/p+1)}(X, -) = z^p(X, E, -).$$

5.1.1. Levine-Serpé tower and slice tower. —

Proposition 5.2. — *The Levine-Serpé tower for $X \times_k K$ is the same with the homotopy convieau tower for $E(X)$ where $X \in \mathbf{Sm}/k$ and G acts on $X \times_k K$ via its action on K.*

Proof. — Since $(|G|, \mathrm{char}(k)) = 1$, the finite field extension K/k is Galois and the obvious map

$$S_X^{(p)}(r) \to S_{G, X \times_k K}^{(p)}(r)$$

is a bijection. We can therefore identify

$$E^{(p)}(X, -) = K^{(p)}(G, X \times_k K, -) \tag{38}$$
$$E^{(p/p+1)}(X, -) = K^{(p/p+1)}(G, X \times_k K, -) \tag{39}$$
$$z^p(X, E, -) = z^p(G, X \times_k K, -). \tag{40}$$

Moreover, the cycle class maps (7) and (17) are compatible with these identifications □

By comparison between the slice and homotopy coniveau tower of E (Theorem 1.5), we have

Theorem 5.3. — *The Levine-Serpé tower for $X \times_k K$ is equivalent to the slice tower for $E(X)$ where $X \in \mathbf{Sm}/k$ and G acts on $X \times_k K$ via its action on K.*

Corollary 5.4. — *There is a natural isomorphism*

$$H^{2q-p}(X, \mathbb{Z}^{K,G}(q)) \cong CH^q(G, X \times_k K, p)$$

Proof. — By definition

$$z^q(X, p, \mathbb{Z}^{K,G}(q)[2q]) = \bigoplus_{\omega \in X^{(q)}(p)} \mathbb{Z}^{K,G}(k(\omega)) = \bigoplus_{\omega \in X^{(q)}(p)} K_0(G, k(\omega) \otimes K).$$

Using (40), we have identifications

$$z^q(X, \mathbb{Z}^{K,G}(q)[2q], p) = z^q(G, X \times_k K, p)$$

that compatible with face and degeneracy maps. Therefore,

$$CH^q(X, \mathbb{Z}^{K,G}(q)[2q], p) = CH^q(G, X \times_k K, p) \tag{41}$$

By Theorem 4.7, $\mathbb{Z}^{K,G}$ is birational motivic sheaf, hence

$$H^{2q-p}(X, \mathbb{Z}^{K,G}(q)) \cong CH^q(X, \mathbb{Z}^{K,G}(q)[2q], p). \tag{42}$$

(see (1.18)). The identities (41) and (42) yield the result. □

Corollary 5.5. — *There is an isomorphism*

$$s_q(E) \cong \mathrm{EM}_{\mathbb{A}^1}(\mathbb{Z}^{K,G}(q)[2q])$$

5.1.2. Grayson tower and slice tower. — We also have isomorphism

$$\mathbb{Z}^{K,G}(n) \cong \mathbb{Z}^{K,G}_{Gr}(n)$$

in the derived category of Nisnevich sheaf on $\mathbf{Sm}/k$. As a consequence, we have

$$f_m^{mot}(\mathbb{Z}^{K,G}_{Gr}(n)) = \begin{cases} 0, & \text{if } m > n \\ \mathbb{Z}^{K,G}_{Gr}(n), & \text{if } 0 \le m \le n \end{cases}$$

and hence

$$s_m^{mot}(\mathbb{Z}^{K,G}_{Gr}(n)) = \begin{cases} 0, & \text{if } m \neq n \\ \mathbb{Z}^{K,G}_{Gr}(n), & \text{if } m = n \ge 0. \end{cases}$$

For any smooth scheme $X \in \mathbf{Sm}/k$ the Grayson tower has the form

$$\ldots \longrightarrow W^2(X) \longrightarrow W^1(X) \longrightarrow W^0(X) \sim K(G, X \times_k K). \tag{43}$$

with

$$W^n(X) = \Omega^{-n}|K(G, X \times_k K \times \Delta^\bullet, \mathbb{G}_m^{\wedge n})|,$$

If X is affine then the successive homotopy cofibers have forms

$$W^n(X)/W^{n+1}(X) \sim \Omega^{-n}|K_0^{\oplus}(G, X \times_k K \times \Delta^\bullet, \mathbb{G}_m^{\wedge n})|.$$

The tower (43) gives a tower of presheaves of spectra on $\mathbf{Sm}/k$. This yields therefore a tower in $Ho(Pre_{Nis}(\mathbf{Sm}/k))$, the homotopy category of presheaves of spectra on the big Nisnevich site $\mathbf{Sm}/k$,

$$\ldots \to W^{n+1} \to W^n \to \ldots \to W^0 \cong K^{K,G} \tag{44}$$

with successive cones given by the complex of Nisnevich sheaves

$$\Sigma_s^n|K_0^{\oplus}(G, - \times_k K \times \Delta^\bullet, \mathbb{G}_m^{\wedge n})|_{Nis}.$$

In other words,

$$W^n/W^{n+1} \cong \mathcal{H}\mathbb{Z}_{Gr}^{K,G}(n).$$

For presheaves of S^1-spectra E and F we denote by $[E, F]$ the abelian group of morphism in $Ho(Pre_{Nis}(\mathbf{Sm}/k))$ between E and F. For any complex of Nisnevich sheaf of abelian groups M, denote by $\mathcal{H}M$ its Eilenberg-MacLane S^1-spectrum. We have

$$[X_+, \Sigma_s^p \mathcal{H}M] \cong H_{Nis}^p(X, M).$$

The complex of Nisnevich sheaves $\mathbb{Z}_{Gr}^{K,G}(n)$ is homotopy invariant. Moreover, it is strictly homotopy invariant because every homotopy invariant Nisnevich sheaf with transfer over a perfect field is strictly homotopy invariant (see [**Voe00b**]). This implies that the S^1-spectra $\mathcal{H}\mathbb{Z}_{Gr}^{K,G}(n)$ is $\mathbb{A}^1$-local. In this case, any fibrant replacement of $\mathcal{H}\mathbb{Z}_{Gr}^{K,G}(n)$ in $\mathbf{Spt}_{S^1}(k)$ with Nisnevich model structure is motivic fibrant in $\mathcal{SH}_{S^1}(k)$, we have therefore

Lemma 5.6. — $\mathrm{Hom}_{\mathcal{SH}_{S^1}(k)}(X_+, \Sigma_s^p \mathcal{H}(\mathbb{Z}_{Gr}^{K,G}(n))) = H_{Nis}^p(X, \mathbb{Z}_{Gr}^{K,G}(n))$.

Lemma 5.7. — (Brown-Gersten) *For any $E : \mathbf{Sm}/k \to \mathbf{Spt}$ that satisfies homotopy invariant and Nisnevich excision we have*

$$\pi_n(E(X)) = \mathrm{Hom}_{\mathcal{SH}_{S^1}(k)}(X_+, \Sigma_s^n E).$$

Hence $K_n(G, X \times_k K) = H_{Nis}^n(X, W^0) = Hom_{\mathcal{SH}_{S^1}(k)}(X_+, \Sigma_s^n W^0)$

Proposition 5.8. — $W^n \in \Sigma_{\mathbb{P}^1}^n \mathcal{SH}_{S^1}(k)$.

Proof. — Apply the operator f_n to the tower (44) we obtain a tower

$$f_n(W^{i+1}) \to f_n(W^i) \to \ldots \to f_n(W^n)$$

with successive quotient

$$f_n(W^i)/f_n(W^{i+1}) \simeq f_n(W^i/W^{i+1}) \simeq f_n(\mathrm{EM}_{\mathbb{A}^1}(\mathbb{Z}^{Gr}(i))) \simeq \mathrm{EM}_{\mathbb{A}^1}(\mathbb{Z}^{Gr}(i)) \simeq W^i/W^{i+1}$$

for all $i \geq n$.

We have a spectral sequence

$$E_2^{p,q} = \mathrm{Hom}_{\mathcal{SH}_{S^1}(k)}(X_+, \Sigma^{p-q} W^{n-q}/W^{n-q+1}) \Rightarrow \mathrm{Hom}_{\mathcal{SH}_{S^1}(k)}(X_+, \Sigma^{-p-q} W^n). \quad (45)$$

Similarly,

$$E_2^{p,q} = \mathrm{Hom}_{\mathcal{SH}_{S^1}(k)}(X_+, \Sigma^{p-q} W^{n-q}/W^{n-q+1}) \Rightarrow \mathrm{Hom}_{\mathcal{SH}_{S^1}(k)}(X_+, \Sigma^{-p-q} f_n(W^n)). \quad (46)$$

Two spectral sequences are strongly convergent because W^i is $(i-1)$-connected and $f_n W^i$ has the same connectivity as W^i(see [**Mor99**] for more details). It implies that

$$\mathrm{Hom}_{\mathcal{SH}_{S^1}(k)}(X, \Sigma^p W^n) \simeq \mathrm{Hom}_{\mathcal{SH}_{S^1}(k)}(X, \Sigma^p f_n(W^n))$$

for any p and $X \in \mathbf{Sm}/k$. Thus $W^n \simeq f_n W^n \in \Sigma_{\mathbb{P}^1}^n \mathcal{SH}_{S^1}(k)$. □

Theorem 5.9. — *For any $n \geq 0$, the canonical morphism $W^n \to f_n W^0$ is an isomorphism in $\mathcal{SH}_{S^1}(k)$. In other words, the tower* (44) *is equivalent to the slice tower for* W^0.

Proof. — For $n = 0$, the identity $f_0W^0 = W^0$ holds by definition. We will use the induction argument to conclude the statement.

Assume that the statement is true for $n-1$. By Proposition 5.8, the natural morphism $W^n \to W^0$ induces a morphism $W^n \to f_n(W^0)$. By universal property of operators f_n, we have a commutative diagram

$$\begin{tikzcd} W^n \arrow[r] \arrow[d] & W^{n-1} \arrow[d] \\ f_nW^0 \arrow[r] & f_{n-1}W^0. \end{tikzcd}$$

Apply the operator f_n to this diagram, we obtain

$$\begin{tikzcd} f_n(W^n) \arrow[r] \arrow[d] & f_n(W^{n-1}) \arrow[d] \\ f_nW^0 \simeq f_nf_nW^0 \arrow[r] & f_nf_{n-1}W^0 \simeq f_nW^0. \end{tikzcd}$$

The top map is an isomorphism because the homotopy cofiber is

$$f_n(W^{n-1}/W^n) \simeq f_n(\mathrm{EM}_{\mathbb{A}^1}(\mathbb{Z}_{Gr}^{K,G}(n-1))) = 0.$$

The bottom map is an isomorphism by definition. The right hand-side map is an isomorphism by induction. We have therefore that $f_nW^n \to f_nW^0$ is an isomorphism.

We also have a commutative diagram

$$\begin{tikzcd} f_n(W^n) \arrow[r] \arrow[d] & W^n \arrow[d] \\ f_nW^0 \arrow[r] & f_nW^0, \end{tikzcd}$$

that implies $W^n \to f_nW^0$ is an isomorphism. □

Corollary 5.10. — *The Grayson and Levine-Serpé spectral sequences are equivalent for $Y \times_k K$ where $Y \in \mathbf{Sm}/k$ is a semi-local scheme with trivial action.*

Proof. — If Y is a semi-local smooth scheme in $\mathbf{Sm}/k$ then

$$H^p(C^*(K_0^{\oplus}(G, - \times_k K, \mathbb{G}_m^{\wedge n}))[-n](Y)) \cong H^p_{Nis}(Y, \mathbb{Z}_{Gr}^{K,G}(n)) \tag{47}$$

(see [**Sus03**, Proposition 1.7]). The left hand-side is the equivariant Grayson cohomology group for $Y \times_k K$. The right hand-side is the equivariant higher Chow group for $Y \times_k K$. □

5.2. Comparision in $\mathcal{SH}(k)$

In this section, we will assume that k is an arbitrary perfect field.

5.2.1. Homotopy coniveau for $\mathbb{P}^1$-Ω-spectra. —

Definition 5.11. — *A $\mathbb{P}^1$-Ω-spectrum $\mathcal{E}$ over k is given by:*

(i) *a sequence $(E_0, E_1, \ldots)$, where each $E_i \in \mathbf{Spt}(k)$ is a homotopy invariant presheaf satisfying Nisnevich excision.*

(ii) *weak equivalences $\epsilon_n : E_n \to \Omega_{\mathbb{P}^1} E_{n+1}$ in $\mathbf{Spt}(k)$, $n = 0, 1, \ldots$.*

A map between two $\mathbb{P}^1$-Ω-spectra is a sequence of maps respecting the ϵ_*.

For any $\mathbb{P}^1$-Ω-spectrum $\mathcal{E} = ((E_0, E_1, \ldots), \epsilon_*)$ and an integer p, set

$$\phi_p \mathcal{E} := ((E_0^{(p)}, E_1^{(p+1)}, \ldots), \phi_p(\epsilon_*))$$

where the maps $\phi_p(\epsilon_n)$ are given by the de-looping weak equivalences

$$(E_n)^{(p+n)} \xrightarrow{\epsilon_n^{(p+n)}} (\Omega_{\mathbb{P}^1} E_{n+1})^{(p+n)} \to \Omega_{\mathbb{P}^1}(E_{n+1})^{(n+p+1)}.$$

The natural maps $E_n^{(p+n)} \to E_n$ define a map of $\mathbb{P}^1$-Ω-spectra

$$\phi_p \mathcal{E} \to \mathcal{E}$$

with remark that $E^{(n)} = E^{(0)}$ for $n \leq 0$. Thus we have a tower of $\mathbb{P}^1$-Ω-spectra

$$\ldots \to \phi_{p+1}\mathcal{E} \to \phi_p \mathcal{E} \to \ldots \to \phi_0 \mathcal{E} \to \phi_{-1}\mathcal{E} \to \ldots \to \mathcal{E}. \tag{48}$$

We write $\phi_{p/p+n}\mathcal{E}$ for the cofiber of $\phi_{p+n}\mathcal{E} \to \phi_p \mathcal{E}$ and $\sigma_p \mathcal{E}$ for $\phi_{p/p+1}\mathcal{E}$.

5.2.2. Slice tower in $\mathcal{SH}(k)$. — Let $\mathcal{SH}^{eff}(k)$ be the smallest localizing subcategory of $\mathcal{SH}(k)$ containing all suspension spectra $\Sigma^\infty_{\mathbb{P}^1} X_+$ with $X \in \mathbf{Sm}/k$. Equivalently, $\mathcal{SH}^{eff}(k)$ is the smallest localizing subcategory containing all the $\mathbb{P}^1$-suspension spectra $\Sigma^\infty_{\mathbb{P}^1} E$ with $E \in \mathcal{SH}_{S^1}(k)$. For each integer p let $\Sigma^p_{\mathbb{P}^1}\mathcal{SH}(k)$ denote the smallest localizing subcategory of $\mathcal{SH}(k)$ containing all the $\mathbb{P}^1$-spectra $\Sigma^p_{\mathbb{P}^1}\mathcal{E}$ for $\mathcal{E} \in \mathcal{SH}^{eff}(k)$. By Brown's representability theorem for triangulated categories, the inclusion

$$i_p : \Sigma^p_{\mathbb{P}^1}\mathcal{SH}^{eff}(k) \to \mathcal{SH}(k)$$

admits the right adjoint

$$r_p : \mathcal{SH}(k) \to \Sigma^p_{\mathbb{P}^1}\mathcal{SH}^{eff}(k).$$

Let $f_p := i_p \circ r_p : \mathcal{SH}(k) \to \mathcal{SH}(k)$ then for any $\mathcal{E} \in \mathcal{SH}(k)$ one has the *slice tower*

$$\ldots \to f_{p+1}\mathcal{E} \to f_p \mathcal{E} \to \ldots \to f_0 \mathcal{E} \to f_{-1}\mathcal{E} \to \ldots \to \mathcal{E} \tag{49}$$

which is functorial in $\mathcal{E}$. The map $f_p\mathcal{E} \to \mathcal{E}$ is universal for maps $\mathcal{F} \to \mathcal{E}$, $\mathcal{F} \in \Sigma^p_{\mathbb{P}^1}\mathcal{SH}^{eff}(k)$ (see Section 1.1.1).

Similar to the case of S^1-spectra, one has a identification between the slice tower and homotopy coniveau tower in $\mathcal{SH}(k)$

Proposition 5.12. — ([**Lev08**, Theorem 9.0.3]) *Let k be a perfect field. For $\mathcal{E} \in \mathcal{SH}(k)$, $\phi_p\mathcal{E}$ is in $\Sigma^p_{\mathbb{P}^1}\mathcal{SH}^{eff}(k)$, and the map $\phi_p\mathcal{E} \to f_p\mathcal{E}$ induced from $\phi_p\mathcal{E} \to \mathcal{E}$ is an isomorphism.*

Remark 5.13. — *For the comparison between the slice tower and homotopy coniveau tower for S^1-spectra (Proposition 5.12), we require k to be an infinite field. The reason is to have the functor $E \to E^{(p)}$ defined for all fibrant E in $\mathbf{Spt}_{S^1}(k)$. For a fibrant (s,p)-spectrum $\mathcal{E} = (E_0, E_1, \ldots)$, the presheaves E_n are all zero-spectra of a fibrant (s,p)-spectrum. In this case, the operation $E_n \to E_n^{(p)}$ is well defined for k finite. Therefore, Proposition 5.12 holds for finite perfect fields.*

Let $\mathcal{Z}^{K,G}$ be the bispectrum

$$\mathcal{Z}^{K,G} := (\mathcal{H}\mathbb{Z}^{K,G}(0), \mathcal{H}\mathbb{Z}^{K,G}(1), \ldots), \epsilon_*).$$

By cancellation theorem, it is a $\mathbb{P}^1$-Ω-spectrum.

The K-theory bispectrum $\mathcal{K}^{K,G} := (K^{K,G}, K^{K,G}, \ldots)$ is an $\mathbb{P}^1$-Ω-spectrum (by projective bundle formula for equivariant K-theory) with $K^{K,G}$ is the zero-spectrum. We have $\Sigma_{\mathbb{P}^1}\mathcal{K}^{K,G} = \mathcal{K}^{K,G}$ and $\sigma_0\mathcal{K}^{K,G} = \mathcal{Z}^{K,G}$.

Theorem 5.14. — *Let k be a perfect field, we have an isomorphism*

$$\mathcal{Z}^{K,G}(p)[2p] \cong \sigma_p\mathcal{K}^{K,G}.$$

As a consequence, the Levine-Serpé spectral sequence for $X \times_k K$ is equivalent to the slice spectral sequence for $\mathcal{K}^{K,G}(X)$.

5.2.3. Grayson tower as a tower of $\mathbb{P}^1$-Ω-spectra. — We consider the following bispectrum

$$\mathcal{W} := ((W^0, W^1, \ldots), \epsilon_*)$$

where W^i is defined as (44) and the bonding maps

$$\epsilon_n : W^n \to \Omega_{\mathbb{P}^1} W^{n+1} \tag{50}$$

are constructed as follow: There is a natural morphism

$$K(G, X, \mathbb{G}^{\wedge n}) \to K(G, X \times \mathbb{G}_m, \mathbb{G}_m^{\wedge n} \times \mathbb{G}_m)$$

which induces a morphism

$$K(G, X, \mathbb{G}^{\wedge n}) \to K(G, X \wedge \mathbb{G}_m, \mathbb{G}_m^{\wedge n} \wedge \mathbb{G}_m) = K(G, X \wedge \mathbb{G}_m, \mathbb{G}_m^{\wedge n+1}).$$

Therefore, we have a morphism of simplicial spectra

$$K(G, X \times \Delta^\bullet, \mathbb{G}_m^{\wedge n}) \to K(G, X \wedge \mathbb{G}_m \times \Delta^\bullet, \mathbb{G}_m^{\wedge n+1})$$

hence a morphism

$$\Omega_s^{-n} K(G, X \times \Delta^\bullet, \mathbb{G}_m^{\wedge n}) \to \Omega_s^{-n}\Omega_{\mathbb{P}^1} K(G, X \times \Delta^\bullet, \mathbb{G}_m^{\wedge n+1})$$

Using the identification $\Omega_{\mathbb{G}_m}\Omega_s = \Omega_{\mathbb{P}^1}$ in $\mathcal{SH}_{S^1}(k)$, the we obtain the desired morphisms $\epsilon_n : W^n \to \Omega_{\mathbb{P}^1} W^{n+1}$.

Lemma 5.15. — *$\mathcal{W}$ is a $\mathbb{P}^1$-Ω-spectrum on $\mathbf{Sm}/k$.*

Proof. — Each presheaf W^n is homotopy invariant because $H^p(X, \mathbb{Z}_{Gr}^{K,G}(n+i))$ is homotopy invariant and the spectral sequence induced from (44) is strongly convergent.

The bonding map $\epsilon_n : W^n \to \Omega_{\mathbb{P}^1} W^{n+1}$ is weak equivalent because the cancellation isomorphisms

$$H^p(X, \mathbb{Z}_{Gr}^{K,G}(q)) \cong H^{p+1}(X, \mathbb{Z}_{Gr}^{K,G}(q+1))$$

are compatible with the Grayson spectral sequence, hence the maps ϵ_n are isomorphisms. $\square$

The tower (44) gives a tower of bispectra

$$\ldots \to \Sigma_{\mathbb{P}^1}^{n+1} \mathcal{W} \to \Sigma_{\mathbb{P}^1}^{n} \mathcal{W} \to \ldots \to \mathcal{W} \tag{51}$$

with successive cones

$$\mathrm{Cone}(\Sigma_{\mathbb{P}^1}^{n+1} \mathcal{W} \to \Sigma_{\mathbb{P}^1}^{n} \mathcal{W}) \cong \Sigma_{\mathbb{P}^1}^{n} \mathcal{Z}^{K,G}.$$

Lemma 5.16. — *The spectral sequence induced from* (51) *is equivalent to the Grayson spectral sequence induced from* (44).

Proof. — It comes from the fact that Grayson cohomology satisfies homotopy invariance and cancellation. For a general argument, see [**GP12**, Theorem 4.7]. $\square$

Lemma 5.17. — $\mathcal{W}$ *is in* $\mathcal{SH}^{eff}(k)$.

Proof. — It comes from Proposition 5.8 and Lemma 5.15. $\square$

Theorem 5.18. — *Let k be a perfect field, the tower* (51) *of bispectra in* $\mathcal{SH}(k)$

$$\ldots \to \Sigma_{\mathbb{P}^1}^{n+1} \mathcal{W} \to \Sigma_{\mathbb{P}^1}^{n} \mathcal{W} \to \ldots \to \mathcal{W}$$

is isomorphic to the tower

$$\ldots \to f_{n+1}(\mathcal{K}) \to f_n(\mathcal{K}) \to \ldots \to f_0(\mathcal{K}).$$

Proof. — Since $\mathcal{W} \in SH^{eff}(k)$, the obvious morphism $\mathcal{W} \to \mathcal{K}$ induces a morphism

$$\mathcal{W} \to f_0 \mathcal{K}.$$

For any $X \in \mathbf{Sm}/k$, we have

$$Hom_{\mathcal{SH}(k)}(X_+, \Sigma_s^p f_0(\mathcal{K})) \cong Hom_{\mathcal{SH}(k)}(X_+, \Sigma_s^p \mathcal{W}) \cong K_p(X).$$

Therefore, $\mathcal{W} \cong f_0(\mathcal{K})$.

Assume that we have constructed an isomorphism $\theta_n : \Sigma_{\mathbb{P}^1}^n \to f_n(\mathcal{K})$. Since $\Sigma_{\mathbb{P}^1}^{n+1} \mathcal{W} \in \Sigma_{\mathbb{P}^1}^{n+1} \mathcal{SH}^{eff}(k)$, there is an unique morphism

$$\theta_{n+1} : \Sigma_{\mathbb{P}^1}^{n+1} \mathcal{W} \to f_{n_1}(\mathcal{K})$$

making the diagram

$$\begin{tikzcd}
\Sigma_{\mathbb{P}^1}^{n+1} \mathcal{W} \arrow[r] \arrow[d, "\theta_{n+1}"] & \Sigma_{\mathbb{P}^1}^{n} \mathcal{W} \arrow[d, "\theta_n"] \\
f_{n+1}(\mathcal{K}) \arrow[r] & f_n(\mathcal{K}).
\end{tikzcd}$$

commute. We will show that θ_{n+1} is an isomorphism in $\mathcal{SH}(k)$ Applying the operation f_{n+1} to this diagram we have the commutative diagram

$$\begin{tikzcd}
f_{n+1}\Sigma_{\mathbb{P}^1}^{n+1}\mathcal{W} \arrow[r] \arrow[d, "f_{n+1}\theta_{n+1}"] & f_{n+1}\Sigma_{\mathbb{P}^1}^{n}\mathcal{W} \arrow[d, "f_{n+1}\theta_n"] \\
f_{n+1}f_{n+1}(\mathcal{K}) \arrow[r] & f_{n+1}f_n(\mathcal{K}).
\end{tikzcd}$$

The bottom arrow is an isomorphism by definition. The map $f_{n+1}(\theta_n)$ is an isomorphism by assumption. The top arrow is an isomorphism because the cone is zero

$$\mathrm{Cone}(f_{n+1}\Sigma_{\mathbb{P}^1}^{n+1}\mathcal{W} \to f_{n+1}\Sigma_{\mathbb{P}^1}^{n}\mathcal{W}) \cong f_{n+1}(\Sigma_{\mathbb{P}^1}^{n}\mathcal{Z}^{K,G}) = 0.$$

Therefore $f_{n+1}(\theta_{n+1})$ is an isomorphism. Since $\mathcal{W} \cong f_0\mathcal{K}$ belongs to $\mathcal{SH}^{eff}(k)$, we have $\Sigma_{\mathbb{P}^1}^{n+1}\mathcal{SH}^{eff}(k)$. This implies that $\theta_{n+1} = f_{n+1}(\theta_{n+1})$, hence θ_{n+1} is an isomorphism. □

5.3. Conclusion

Unlike the ordinary case where all the motivic spectral sequences are equivalent for smooth semi-local schemes of finite type over k, the Levine-Serpé and equivariant Grayson spectral sequences are different for smooth semi-local G-schemes of finite type over a field in general.

Let $\mathbb{A}^1_0$ be the smooth local scheme obtained by localizing the affine line $\mathbb{A}^1 = \mathrm{Spec}k[t]$ at the origin $0 \in \mathbb{A}^1$. The group $G = \mathbb{Z}/2$ acts on $\mathbb{A}^1$ by $t \mapsto -t$ that induces an action on $\mathbb{A}^1_0$. Since $\mathbb{Z}/2$ acts trivially on $\mathrm{Spec}k$, we have

$$K_0(\mathbb{Z}/2, \mathrm{Spec}k) = \mathbb{Z} \oplus \mathbb{Z}.$$

Consider the commutative diagram

$$\begin{tikzcd}
 & \mathbb{A}^1 \arrow[dr, "\pi"] & \\
\mathbb{A}^1_0 \arrow[ur, "i"] \arrow[rr, "\pi_0"] & & \mathrm{Spec}k
\end{tikzcd} \tag{52}$$

with obvious morphisms. Apply the $K_0(\mathbb{Z}/2, -)$ functor to (52), with remark that $i^* : K_0(\mathbb{Z}/2, \mathbb{A}^1) \to K_0(\mathbb{Z}/2, \mathbb{A}^1_0)$ is surjective by localization property and $\pi^* : K_0(\mathbb{Z}/2, \mathrm{Spec}k) \to K_0(\mathbb{Z}/2, \mathbb{A}^1)$ is an isomorphism by homotopy invariance, we have $\pi_0^* : K_0(\mathbb{Z}/2, \mathrm{Spec}k) \to K_0(\mathbb{Z}/2, \mathbb{A}^1_0)$ is surjective.

Let $i_0 : \mathrm{Spec}k \to \mathbb{A}^1_0$ be the "origin" inclusion then

$$\pi_0 \circ i_0 : \mathrm{Spec}k \xrightarrow{i_0} \mathbb{A}^1_0 \xrightarrow{\pi_0} \mathrm{Spec}k$$

is the identity map. Therefore, $i_0^* \circ \pi_0^* = \mathrm{id}_{K_0(\mathbb{Z}/2,\mathrm{Spec}k)}$ which implies that π_0^* is injective map. Hence π_0^* is an isomorphism and we have

$$\pi_0^* : K_0(\mathbb{Z}/2, \mathbb{A}^1_0) \cong K_0(\mathbb{Z}/2, \mathrm{Spec}k) = \mathbb{Z} \oplus \mathbb{Z}.$$

The fraction field of $\mathbb{A}^1_0$ is $k(t)$ where $\mathbb{Z}/2$ acts by sending t to $-t$, hence

$$K_0(\mathbb{Z}/2, k(t)) \cong K_0(k(t^2)) = \mathbb{Z}.$$

We have

$$H^0_{Gr}(\mathbb{Z}/2, \mathbb{A}^1_0, 0) = K_0(\mathbb{Z}/2, \mathbb{A}^1_0) = \mathbb{Z} \oplus \mathbb{Z}.$$

However,

$$CH^0(\mathbb{Z}/2, \mathbb{A}^1_0, 0) = K^0(\mathbb{Z}/2, \mathrm{Spec}k) = \mathbb{Z}.$$

5.4. Further remarks

In any equivariant motivic homotopy category that we hope to construct, there are many *sphere-like objects* appear: the *representation spheres*. For any k-linear representation V of G, let $\mathbb{P}(V)$ be the corresponding projective spaces, we definite the sphere S^V as the *one-point compactification* of V, namely

$$S^V := \mathbb{P}(V \oplus \mathbb{A}^1)/\mathbb{P}(V). \tag{53}$$

The action of G on V induces an action on S^V. The motivic sphere $\mathbb{G}_m$ (or $\mathbb{P}^1$) appears in this way with trivial action. If $(|G|, \mathrm{char}k) = 1$, then every representation decomposes as a direct sum of irreducible representations. In this case, every representation sphere is a smash product of spheres of irreducible representations. In other world, sphere-like objects are indexed by irreducible representations of G.

To construct a G-equivariant motivic unstable homotopy category, we can use the method presented in [**Del09**]. The method for stabilizing can be taken from [**Jar00**], where we can mimic most the construction of the motivic stable homotopy category $\mathcal{SH}(k)$ to obtain the category of S^V-spectra. However, there are plenty of spheres that we can choose to stabilize and, of course, the resulting categories might be very different.

When we fix a representation V, we are able to construct the slice tower (or Postnikov tower) for S^V-spectra. If the G-equivariant motivic homotopy category is nice enough such that the equivariant algebraic K-theory is representable by a spectrum $\mathcal{K}_G$, we can consider the (S^V-) slice tower for this object. This task is obviously difficult. We guess that the Levine-Serpé's and Grayson's tower correspond to two different (S^V-) slice towers for $\mathcal{K}_G$.

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